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Fluctuations of inflationary magnetogenesis

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Abstract

This analysis aims at exploring what can be said about the growth rate of magnetized inhomogeneities under two concurrent hypotheses: a phase of quasi-de Sitter dynamics driven by a single inflaton field and the simultaneous presence of a spectator field coupled to gravity and to the gauge sector. Instead of invoking ad hoc correlations between the various components, the system of scalar inhomogeneities is diagonalized in terms of two gauge-invariant quasi-normal modes whose weighted sum gives the curvature perturbations on comoving orthogonal hypersurfaces. The predominance of the conventional adiabatic scalar mode implies that the growth rate of magnetized inhomogeneities must not exceed 2.2 in Hubble units if the conventional inflationary phase is to last about 70 efolds and for a range of slow roll parameters between 0.1 and 0.001. Longer and shorter durations of the quasi-de Sitter stage lead, respectively, either to tighter or to looser bounds which are anyway more constraining than the standard backreaction demands imposed on the gauge sector. Since a critical growth rate of order 2 leads to a quasi-flat magnetic energy spectrum, the upper bounds on the growth rate imply a lower bound on the magnetic spectral index. The advantages of the uniform curvature gauge are emphasized and specifically exploited throughout the treatment of the multicomponent system characterizing this class of problems.

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1 Formulation of the problem

According to a recurrent theme of speculations, large-scale magnetic fields could be generated in the early Universe [1, 2, 3, 4, 5, 6]. The curvature perturbations evolving for typical length-scales larger than the Hubble radius can thus be magnetized with a mechanism bearing some resemblance to a pristine non-adiabatic pressure fluctuation. This observation has been used some time ago to argue that the evolution of curvature perturbations constrains the magnetic power spectra [7]. In the present paper the same logic explored in [7], i.e. the predominance of the adiabatic mode over the gauge contributions, will be used to analyze consistently the fluctuations of inflationary magnetogenesis and derive different constraints on the growth rate of the corresponding inhomogeneities.

The fate of magnetized scalar modes during diverse dynamical regimes can be followed through a gauge-invariant variable, conventionally denoted by ζ, describing either the curvature perturbations on the hypersurface where the energy density is uniform or, complementarily, the density contrast on uniform curvature hypersurfaces. The latter interpretation becomes physically appealing and mathematically simpler in the so-called uniform curvature gauge which has been discussed in different contexts [8, 9, 10, 11]. Since ζ is ultimately gauge-invariant its evolution can be studied in any gauge and the result of [7], derived originally in the uniform curvature gauge [12], can be confirmed in different coordinate systems and in different dynamical situations [13] [2]; neglecting electric fields and Ohmic currents the evolution equation of ζ is:

$$\zeta' = -\frac{\mathcal{H}}{\rho_{\rm t}(1+w_{\rm t})}\delta p_{\rm nad} + \frac{\mathcal{H}(3c_{\rm st}^2 - 1)}{3\rho_{\rm t}(1+w_{\rm t})}\delta\rho_{\rm B} - \frac{\theta_{\rm t}}{3}, \tag{1.1}$$

where $\delta p_{\rm nad}(\vec{x},\tau)$ accounts for the non-adiabatic pressure inhomogeneities; $\delta\rho_{\rm B}(\vec{x},\tau)$ is the fluctuation of the magnetic energy density and $\theta_{\rm t} = \vec{\nabla}\cdot\vec{v}_{\rm t}$ is the divergence of the total velocity field. Barring for a possible contribution of the total velocity field[3] and in the absence of entropic modes (i.e. $\delta p_{\rm nad} = 0$) the solution of Eq. (1.1) is in fact a functional of the total barotropic index $w_{\rm t} = p_{\rm t}/\rho_{\rm t}$ and of the total sound speed $c_{\rm st}^2 = p_{\rm t}'/\rho_{\rm t}'$. Denoting with $\zeta_*(\vec{x})$ the conventional adiabatic mode, the full solution of Eq. (1.1) becomes:

$$\zeta(\vec{x},a,a_*) = \zeta_*(\vec{x}) + \int_{a_*}^{a} \frac{(3c_{\rm st}^2(b) - 1)}{3\rho_{\rm t}(b)[1+w_{\rm t}(b)]}\delta\rho_{\rm B}(\vec{x},b)\, d\ln b, \tag{1.2}$$

where the integration variable is provided directly by the scale factor[4].

[2] As usual the prime denotes the derivation with respect to the conformal time coordinate τ and $\mathcal{H} = a'/a$ where a is the scale factor of a conformally flat metric of Friedmann-Robertson-Walker type.

[3] This term is subleading for wavelengths larger than the Hubble radius at the corresponding epoch.

[4] Equation (1.1) corresponds exactly to Eq. (2.15) of Ref. [7]. The same equation has been used [12] to deduce the initial conditions of the Cosmic Microwave Background (CMB) anisotropies in the presence of postinflationary magnetic fields characterizing the so called magnetized adiabatic mode. Equation (1.1) has been later generalized to the case when Ohmic currents are present and also in the framework of the

In the uniform curvature gauge [8, 9, 10, 11], Eq. (1.1) stems directly from the covariant conservation of the total energy-momentum tensor on uniform curvature hypersurfaces:

$$\delta\rho_{\mathrm{t}}' + 3\mathcal{H}(\delta\rho_{\mathrm{t}} + \delta p_{\mathrm{t}}) + (p_{\mathrm{t}} + \rho_{\mathrm{t}})\theta_{\mathrm{t}} = 0, \qquad \zeta = \frac{(\delta\rho_{\mathrm{t}} + \delta\rho_{\mathrm{B}})}{3(1 + w_{\mathrm{t}})\rho_{\mathrm{t}}}, \tag{1.3}$$

where, by definition[5], $\delta p_{\mathrm{t}} = c_{\mathrm{st}}^2 \delta\rho_{\mathrm{t}} + \delta p_{\mathrm{nad}}$. The covariant conservation of the total energy-momentum tensor also implies the adiabatic suppression of $\delta\rho_{\mathrm{B}}(\vec{x}, \tau)$ redshifting as a^{-4}. A more general derivation of Eq. (1.1) including Ohmic currents and energy flow is swiftly outlined in Eq. (A.28) of appendix A.

The strategy leading to Eqs. (1.1)–(1.3) could be rigidly translated, at first sight, directly during the inflationary stage of expansion. It might then seem plausible to keep the whole logic untouched but to concoct specific modifications of the evolution of $\delta\rho_{\mathrm{B}}$ modeling, via an appropriate rate of increase, the growth of $\delta\rho_{\mathrm{B}}$ during inflation when the relevant wavelengths of the corresponding fluctuations are larger than the Hubble radius. A candidate equation describing the amplification of the magnetic inhomogeneities is, for instance,

$$\delta\rho_{\mathrm{B}}' + 4\mathcal{H}\delta\rho_{\mathrm{B}} = 2\mathcal{F}\delta\rho_{\mathrm{B}}, \tag{1.4}$$

where $2\mathcal{F}$ denotes the rate of increase of the magnetic energy density which is twice the growth rate of the magnetic field itself. Barring for the presence of Ohmic currents and electric fields, Eq. (1.4) partially accounts for the effect of superadiabatic amplification of the magnetic fields but disrupts the covariant conservation of the total system. This means that the evolution equation for ζ is no longer valid. A compensating term can be added at the right hand side of Eq. (1.3) but this has different drawbacks since the evolution equations derived from the covariant conservation of the total energy-momentum tensor will be no longer compatible with the remaining perturbed Einstein equations.

If the dynamics of the inflationary magnetogenesis is not taken into account specifically, the evolution of the whole system turns out to be inconsistent because of the lack of covariant conservation of the total energy-momentum tensor. The first mandatory step for any analysis involving the fluctuations of inflationary magnetogenesis is to posit a perfectible framework where magnetic fields are amplified, the Bianchi identities are satisfied and the inflationary dynamics is satisfactorily implemented. We suggest that the dynamics of magnetized inhomogeneities can be consistently scrutinized in the following system:

$$\mathcal{G}_{\mu}^{\nu} = 8\pi G\Big[T_{\mu}^{\nu}(\varphi) + T_{\mu}^{\nu}(\sigma) + \mathcal{T}_{\mu}^{\nu}(p,\rho) + \mathcal{Z}_{\mu}^{\nu}(Y)\Big], \tag{1.5}$$

gradient expansion (see, respectively, the first and second paper of [13]). Exactly the same equation (1.1) has been applied in Ref. [14] with virtually the same purpose of deriving a bound connecting the amplitude of the adiabatic mode and the strength of the magnetic field.

[5]The total pressure can fluctuate either because of a change in the energy density (when the specific entropy is unperturbed) or because of a change in the specific entropy of the system (when the energy density is unperturbed).

$\mathcal{G}_{\mu}^{\nu}$ denotes the Einstein tensor while $T_{\mu}^{\nu}(\varphi)$ is the energy-momentum tensor of the inflaton φ; $T_{\mu}^{\nu}(\sigma)$ is the energy-momentum tensor of a spectator field σ and $\mathcal{T}_{\mu}^{\nu}(p,\rho)$ is the energy-momentum tensors of the total fluid sources while $\mathcal{Z}_{\mu}^{\nu}(Y)$ is the energy-momentum tensor of the gauge fields. The explicit coupling to the spectator or to the inflaton fields leads to the covariant non-conservation of $\mathcal{Z}_{\mu}^{\nu}$

$$\nabla_{\mu}\mathcal{Z}_{\nu}^{\mu} = \frac{\partial_{\nu}\lambda}{16\pi}\, Y_{\alpha\beta}\, Y^{\alpha\beta} + j^{\alpha}\, Y_{\alpha\nu}, \tag{1.6}$$

where $Y_{\alpha\beta}$ is the gauge field strength, j_{α} is the four-current and $\lambda(x)$ is a function parametrizing the coupling between the gauge fields and the spectator field σ. For sake of generality we shall also consider the possibility the coupling will depend both on σ and φ so that $\lambda = \lambda(\sigma,\varphi)$. The covariant non-conservation of $\mathcal{Z}_{\nu}^{\mu}$ is compensated by the covariant non-conservation of the other energy-momentum tensors:

$$\nabla_{\mu}T_{\nu}^{\mu}(\varphi) = -\frac{\partial_{\nu}\varphi}{16\pi}\,\frac{\partial\lambda}{\partial\varphi}\, Y_{\alpha\beta}\, Y^{\alpha\beta}, \tag{1.7}$$

$$\nabla_{\mu}T_{\nu}^{\mu}(\sigma) = -\frac{\partial_{\nu}\sigma}{16\pi}\,\frac{\partial\lambda}{\partial\sigma}\, Y_{\alpha\beta}\, Y^{\alpha\beta}, \tag{1.8}$$

$$\nabla_{\mu}\mathcal{T}_{\nu}^{\mu} = -j^{\alpha}\, Y_{\alpha\nu}. \tag{1.9}$$

Equation (1.5) captures a class of magnetogenesis scenarios studied along different perspectives through the years and some of the possibilities will now be recalled. In general terms $\lambda = \lambda[\varphi(x),\, \sigma(x),\, ...]$ may be a functional of various scalar degrees of freedom such as the inflaton φ [15], the dilaton [16], a dynamic gauge coupling [17, 18] (see also [19, 20]). The field λ can be a functional of a spectator field σ, [21, 22] (see also [23, 24]) evolving during the inflationary phase; in this case there is no connection between the evolution of λ and the gauge coupling. Some of these possibilities can be realized in the case of bouncing models [16], some other are compatible with the standard inflationary paradigm [15, 17, 18, 21]. It is finally worth recalling a recent observation: the initial conditions of inflationary magnetogenesis may be conducting [27] since the Ohmic currents present during the preinflationary dynamics are not damped by expansion due to the Weyl invariance of the electromagnetic sources.

A perturbative treatment of the fluctuations of inflationary magnetogenesis in a consistent dynamical framework encompassing the inhomogeneities of the inflaton, of the spectator field, of the growth factor and, last but not least, of the relevant plasma variables will now be presented. This analysis is lacking and it is mandatory if the principle of predominance of the adiabatic mode, spelled out in of [7], is to be enforced during the inflationary phase. The tools developed in this paper will allow for an accurate constraint involving simultaneously the slow roll parameters, the total number of inflationary efolds and the total rate of increase which can be defined, for the present purposes, directly from Eqs. (1.6) as $\mathcal{F} = \partial_{\tau}\sqrt{\lambda}/\sqrt{\lambda}$. Inspired by the analysis of [7] the pivotal variables for the evolution of the gauge sector will not be the gauge fields but rather the components of the energy-momentum tensor. This

strategy together with the gauge choice mentioned above will allow for a swifter calculation of the primary and secondary curvature perturbations induced by the inflaton field and by the spectator field.

The layout of this paper is the following. In Sec. 2 the general equations of the system will be discussed and the main notations specified. In Sec. 3 the description of stochastic averages will be introduced with the aim of reducing the evolution of the system to the evolution of the components of the energy-momentum tensor of the gauge field inhomogeneities. In Sec. 4 the quasi-normal modes of inflationary magnetogenesis will be discussed in general terms. In Sec. 5 the magnetized power spectra of the scalar modes will be computed while in Sec. 6 the bounds on the growth rate of the magnetic energy density will be derived. To avoid lengthy digressions various technical details have been collected in the appendices: in appendix A the evolution equations of the system have been explicitly derived in the uniform curvature gauge systematically used in the analysis; in appendix B the second-order correlations of the electric and magnetic fields have been specifically computed and analyzed.

2 Basic equations and definitions

The system of equations swiftly outlined in Eqs. (1.6) and (1.7)–(1.9) can be illustrated by specifying the actions of the different contributions:

$$S_{\rm tot} = S_{\rm gravity} + S_{\varphi} + S_{\sigma} + S_{\rm em} + S_{\rm fluid}, \tag{2.1}$$

where the first three terms of Eq. (2.1) are given by:

$$S_{\rm gravity} + S_{\varphi} = \int d^4x \sqrt{-g}\,\biggl[-\frac{1}{2\ell_{\rm P}^2} R + \frac{1}{2} g^{\alpha\beta}\partial_{\alpha}\varphi\partial_{\beta}\varphi - V(\varphi)\biggr], \tag{2.2}$$

$$S_{\sigma} = \int d^4x \sqrt{-g}\,\biggl[\frac{1}{2} g^{\alpha\beta}\partial_{\alpha}\sigma\partial_{\beta}\sigma - W(\sigma)\biggr], \tag{2.3}$$

$$S_{\rm em} = -\frac{1}{16\pi}\int d^4x \sqrt{-g}\,\lambda(\varphi,\sigma)\, Y_{\mu\nu}\, Y^{\mu\nu} - \int d^4x \sqrt{-g}\, j_{\mu}\, Y^{\mu} + S_{(+)} + S_{(-)}. \tag{2.4}$$

In Eqs. (2.2)–(2.3), $V(\varphi)$ and $W(\sigma)$ denote, respectively, the potentials of the inflaton field and of the spectator field. In Eq. (2.4) $j_{\mu} = j_{\mu}^{(+)} - j_{\mu}^{(-)}$ is the total current; $S_{(\pm)}$ are the actions of the charged species while the last term of Eq. (2.1) (parametrized via a barotropic fluid) can be important either at the onset of inflation (for conducting initial conditions [27]) or during the postinflationary phase. The notations for the Planck length and for the Planck mass in units $\hbar = c = \kappa_{\rm B} = 1$ are as follows

$$\ell_{\rm P}^2 = 8\,\pi\, G = \frac{8\pi}{M_{\rm P}^2} = \frac{1}{\overline{M}_{\rm P}^2}, \tag{2.5}$$

where $M_{\rm P} = G^{-1/2} = 1.22 \times 10^{19}\,{\rm GeV}$. On top of Eq. (1.5), The equations of motion of the various fields appearing in Eqs. (2.1) and (2.4) are given by Eq. (1.5) supplemented by the

following three equations:

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + \frac{\partial V}{\partial\varphi} + \frac{1}{16\pi}\frac{\partial\lambda}{\partial\varphi}Y_{\alpha\beta}Y^{\alpha\beta} = 0, \tag{2.6}$$

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\sigma + \frac{\partial W}{\partial\sigma} + \frac{1}{16\pi}\frac{\partial\lambda}{\partial\sigma}Y_{\alpha\beta}Y^{\alpha\beta} = 0, \tag{2.7}$$

$$\nabla_\alpha\mathcal{T}_\beta^\alpha = 0; \tag{2.8}$$

the explicit forms of the energy-momentum tensors $T_\alpha^\beta(\varphi)$, $T_\alpha^\beta(\sigma)$ and $\mathcal{T}_\alpha^\beta(\rho,\, p)$ are:

$$T_\alpha^\beta(\varphi) = \partial_\alpha\varphi\partial^\beta\varphi - \left[\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right]\delta_\alpha^\beta, \tag{2.9}$$

$$T_\alpha^\beta(\sigma) = \partial_\alpha\sigma\partial^\beta\sigma - \left[\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - W(\sigma)\right]\delta_\alpha^\beta, \tag{2.10}$$

$$\mathcal{T}_\alpha^\beta(\rho,\, p) = (p+\rho)\, u_\alpha u^\beta - p\,\delta_\alpha^\beta, \tag{2.11}$$

$$\mathcal{Z}_\alpha^\beta(Y) = \frac{\lambda}{4\pi}\left[-Y_{\alpha\mu}Y^{\beta\mu} + \frac{1}{4}\delta_\alpha^\beta\, Y_{\mu\nu}\, Y^{\mu\nu}\right], \tag{2.12}$$

where $g^{\alpha\beta}\, u_\alpha\, u_\beta = 1$. By using Eqs. (2.9)–(2.10) and (2.11)–(2.12) the evolution equations for the energy-momentum tensors mentioned in Eqs. (1.7)–(1.9), Eqs. (2.6) and (2.7) can be reproduced bearing in mind the following pair of equations for the gauge fields:

$$\nabla_\alpha\Bigl(\lambda\, Y^{\alpha\beta}\Bigr) = 4\pi j^\beta, \qquad \nabla_\alpha\tilde{Y}^{\alpha\beta} = 0; \tag{2.13}$$

the dual field strength is defined as $\tilde{Y}^{\alpha\beta} = E^{\alpha\beta\mu\nu}Y_{\mu\nu}/2$ in terms of the Levi-Civita tensor density $E^{\alpha\beta\mu\nu} = \epsilon^{\alpha\beta\mu\nu}/\sqrt{-g}$. Note, finally, as already mentioned in Sec. 1 that $\lambda = \lambda(\varphi, \sigma)$ and, consequently, $\partial_\mu\lambda = (\partial_\varphi\lambda\partial_\mu\varphi + \partial_\sigma\lambda\partial_\mu\sigma)$.

2.1 Background evolutions and some approximations

In a conformally flat background of the type $\overline{g}_{\alpha\beta} = a^2(\tau)\eta_{\alpha\beta}$ (where $a(\tau)$ is the scale factor and $\eta_{\alpha\beta}$ is the Minkowski metric), Eqs. (1.5) and (2.6)–(2.7) lead to the following set of equations valid during the inflationary phase

$$3\overline{M}_{\mathrm{P}}^2\,\mathcal{H}^2 = \frac{1}{2}(\varphi^{\prime 2} + \sigma^{\prime 2}) + a^2\, V(\varphi) + a^2\, W(\sigma), \tag{2.14}$$

$$2\overline{M}_{\mathrm{P}}^2\,(\mathcal{H}^2 - \mathcal{H}^\prime) = \varphi^{\prime 2} + \sigma^{\prime 2}, \tag{2.15}$$

$$\varphi^{\prime\prime} + 2\mathcal{H}\varphi^\prime + \frac{\partial V}{\partial\varphi}a^2 = 0, \tag{2.16}$$

$$\sigma^{\prime\prime} + 2\mathcal{H}\sigma^\prime + \frac{\partial W}{\partial\sigma}a^2 = 0. \tag{2.17}$$

As mentioned prior to Eq. (1.1), in Eqs. (2.14)–(2.17) the prime denotes a derivation with respect to the conformal time coordinate τ; furthermore $\mathcal{H} = (\ln a)^\prime = aH$ where $H = \dot{a}/a$

is the conventional Hubble rate and the overdot denotes a derivation with respect to the cosmic time coordinate t.

The slow roll approximation completely defines the evolution during the inflationary phase where the parameters ϵ, η and $\overline{\eta}$ are all much smaller than 1 and eventually get to 1 when inflation ends. The definitions of the slow roll parameters within the notations of this paper are as follows:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\overline{M}_{\rm P}^2}{2}\left(\frac{V_{,\varphi}}{V}\right)^2, \qquad \eta = \frac{\ddot{\varphi}}{H\dot{\varphi}}, \qquad \overline{\eta} = \overline{M}_{\rm P}^2\left(\frac{V_{,\varphi\varphi}}{V}\right), \tag{2.18}$$

note that $V_{,\varphi}$ and $V_{,\varphi\varphi}$ are shorthand notations for the first and second derivatives of the potential $V(\varphi)$ with respect to φ. The slow roll parameters η, $\overline{\eta}$ and ϵ are not independent and their mutual relation, i.e. $\eta = \epsilon - \overline{\eta}$, follows from the slow roll equations written in the cosmic time coordinate:

$$3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0, \quad 3\overline{M}_{\rm P}^2 H^2 = V, \quad 2\overline{M}_{\rm P}^2 \dot{H} = -\dot{\varphi}^2, \tag{2.19}$$

where, by definition of spectator field, we have that $\rho_\sigma \ll \rho_\varphi$ and $\dot{\varphi}^2 \gg \dot{\sigma}^2$ having introduced the energy densities of the inflaton ρ_φ and of the spectator field ρ_σ. In the slow roll approximation and for constant ϵ we have that

$$\mathcal{H} = aH = -\frac{1}{(1-\epsilon)\tau}. \tag{2.20}$$

There are some classes of exact solutions which shall be used in order to test the specific approximations discussed in the second part of this analysis. If both φ and σ have exponential potentials a solution of the system (2.14)–(2.17) subjected to the constraint that $\rho_\sigma \ll \rho_\varphi$ and $\dot{\varphi}^2 \gg \dot{\sigma}^2$ can be written, in cosmic time, as:

$$a(t) = (H_1\, t)^{\alpha}, \qquad \varphi(t) = \sqrt{2\alpha}\,\overline{M}_{\rm P} \ln{(H_1\, t)}, \tag{2.21}$$

$$V(\varphi) = \overline{M}_{\rm P}^2\, H_1^2 (3\alpha^2 - \alpha) \exp\left[-\sqrt{\frac{2}{\alpha}}\frac{\varphi}{\overline{M}_{\rm P}}\right], \tag{2.22}$$

$$\sigma(t) = 2M \ln{(Mt)}, \qquad W(\sigma) = 2(3\alpha - 1) M^4 \exp\left[-\frac{\sigma}{M}\right], \tag{2.23}$$

with $M \ll \overline{M}_{\rm P}$ and $\alpha \gg 1$ so that $\epsilon = \eta = 1/\alpha \ll 1$. In conformal time the corresponding scale factor becomes:

$$a(\tau) = \left(-\frac{\tau}{\tau_1}\right)^{-\beta}, \qquad \beta = \frac{\alpha}{\alpha - 1}, \tag{2.24}$$

with $\beta \to 1$ in the limit $\alpha \gg 1$ and $\epsilon \ll 1$. In specific models of inflationary evolution, the values of the slow roll parameters, for a given number of efolds, can be related to the properties of the potential. To keep the discussion sufficiently general we shall treat the slow roll parameters and the number of efolds as independent variables; conversely, as already mentioned prior to Eq. (2.20), the slow roll parameters will be taken to be constant implying that the inflationary potentials considered here have a monomial form.

3 Quantum and stochastic descriptions

The amplification of the gauge fields can be described quantum mechanically in terms of the appropriate canonical field operators and of their related mode functions. This description is equivalent to the evolution of the power spectra of the different correlations. The two approaches are related and this observation turns out to be very practical for the present considerations.

3.1 Evolution of the canonical gauge field fluctuations

In the conformally flat background discussed in the previous section, Eq. (2.13) becomes explicit in terms of the canonical electric and magnetic fields diagonalizing the action and the canonical Hamiltonian [27]:

$$\frac{1}{\sqrt{\lambda}}\vec{\nabla}\cdot(\sqrt{\lambda}\,\vec{E}) = 4\pi q(n_{+} - n_{-}), \qquad \sqrt{\lambda}\vec{\nabla}\cdot\left(\frac{\vec{B}}{\sqrt{\lambda}}\right) = 0, \tag{3.1}$$

$$\frac{1}{\sqrt{\lambda}}\vec{\nabla}\times(\sqrt{\lambda}\,\vec{B}) = 4\,\pi\,q(n_{+}\vec{v}_{+} - n_{-}\vec{v}_{-}) + \frac{1}{\sqrt{\lambda}}\frac{\partial}{\partial\tau}(\sqrt{\lambda}\,\vec{E}), \tag{3.2}$$

$$\sqrt{\lambda}\vec{\nabla}\times\left(\frac{\vec{E}}{\sqrt{\lambda}}\right) = -\sqrt{\lambda}\frac{\partial}{\partial\tau}\left(\frac{\vec{B}}{\sqrt{\lambda}}\right), \tag{3.3}$$

where $\vec{E}(\vec{x},\tau)$ and $\vec{B}(\vec{x},\tau)$ are

$$\vec{E} = a^{2}\,\sqrt{\lambda}\;\vec{e}, \qquad \vec{B} = a^{2}\,\sqrt{\lambda}\;\vec{b}. \tag{3.4}$$

The fields $\vec{e}$ and $\vec{b}$ are introduced from the corresponding field strengths, i.e. $Y_{i\,0} = -a^{2}\,e_{i}$ and $Y_{i\,j} = -a^{2}\epsilon_{i\,j\,k}\,b^{k}$. The gauge action is canonical in terms of $\vec{E}$ and $\vec{B}$ and not in terms of $\vec{e}$ and $\vec{b}$. Furthermore the system of Eqs. (3.1)–(3.3), in the absence of electromagnetic sources, is invariant under the generalized duality transformation $\vec{E}\to-\vec{B}$, $\vec{B}\to\vec{E}$ and $\sqrt{\lambda}\to 1/\sqrt{\lambda}$ [25, 26] (see also the second paper quoted in Ref. [27]).

3.2 Evolution of the power spectra

Let us start by recalling the notion of stochastically distributed Fourier modes in the case of the electric and magnetic fields, i.e.

$$\langle B_{i}(\vec{q},\tau)\,B_{j}(\vec{p},\tau)\rangle = \frac{2\pi^{2}}{q^{3}}\,P_{\mathrm{B}}(q,\tau)\,P_{ij}(\hat{q})\,\delta^{(3)}(\vec{q}+\vec{p}), \tag{3.5}$$

$$\langle E_{i}(\vec{q},\tau)\,E_{j}(\vec{p},\tau)\rangle = \frac{2\pi^{2}}{q^{3}}\,P_{\mathrm{E}}(q,\tau)\,P_{ij}(\hat{q})\,\delta^{(3)}(\vec{q}+\vec{p}), \tag{3.6}$$

where $P_{ij}(\hat{q}) = (\delta_{ij} - \hat{q}_{i}\hat{q}_{j})$ (with $\hat{q}_{i} = q_{i}/|\vec{q}|$); the conventions for the Fourier transform are:

$$B_{i}(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^{3}k\,B_{i}(\vec{k},\tau)\,e^{-i\vec{k}\cdot\vec{x}}, \qquad E_{i}(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^{3}k\,E_{i}(\vec{k},\tau)\,e^{-i\vec{k}\cdot\vec{x}}. \tag{3.7}$$

The evolution equations (3.1), (3.2) and (3.3) are equivalent to the following set of equations obeyed by the power spectra of Eqs. (3.5)–(3.6)

$$\frac{\partial P_{\mathrm{B}}}{\partial \tau} = 2\mathcal{F} P_{\mathrm{B}} - q P_{\mathrm{EB}}, \tag{3.8}$$

$$\frac{\partial P_{\mathrm{E}}}{\partial \tau} = -2(\mathcal{F} + 4\pi\sigma_{\mathrm{c}}) P_{\mathrm{E}} + q P_{\mathrm{EB}}, \tag{3.9}$$

$$\frac{\partial P_{\mathrm{EB}}}{\partial \tau} = 2q(P_{\mathrm{B}} - P_{\mathrm{E}}) - 4\pi\sigma_{\mathrm{c}} P_{\mathrm{EB}}, \tag{3.10}$$

where σ_{c} denotes the Ohmic conductivity, $\mathcal{F} = \sqrt{\lambda}'/\sqrt{\lambda}$ is the growth rate and P_{EB} is the cross-correlation spectrum defined implicitly by the following equation

$$\langle \vec{E} \cdot \vec{\nabla} \times \vec{B} \rangle + \langle \vec{B} \cdot \vec{\nabla} \times \vec{E} \rangle = 2 \int dq\, P_{\mathrm{EB}}(q, \tau). \tag{3.11}$$

The cross-correlation spectrum provides the physical difference between a stochastic collection of gauge fields (described by Eqs. (3.5) and (3.6)) and their quantum analog which will be discussed in a moment (see Eqs. (3.17)–(3.17)). Conducting initial conditions [27] correspond, in Eqs. (3.6)–(3.8), to the limit $P_{\mathrm{EB}} \to 0$ where the magnetic fields are amplified and the electric fields suppressed either at the same rate or even exponentially depending on the value of the protoinflationary conductivity. In quantum mechanical terms the canonical normal modes are field operators defined as[6]

$$\hat{B}_i(\vec{x}, \tau) = -\frac{i}{(2\pi)^{3/2}} \epsilon_{mni} \sum_{\alpha} \int d^3k\, k_m\, e_n^{\alpha} \Big[f_k(\tau)\, \hat{a}_{\vec{k},\alpha} e^{-i\vec{k}\cdot\vec{x}} - f_k^{*}(\tau) \hat{a}^{\dagger}_{\vec{k},\alpha} e^{i\vec{k}\cdot\vec{x}} \Big], \tag{3.12}$$

$$\hat{E}_i(\vec{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \sum_{\alpha} \int d^3k\, e_i^{\alpha} \Big[g_k(\tau)\, \hat{a}_{\vec{k},\alpha} e^{-i\vec{k}\cdot\vec{x}} + g_k^{*}(\tau) \hat{a}^{\dagger}_{\vec{k},\alpha} e^{i\vec{k}\cdot\vec{x}} \Big], \tag{3.13}$$

where the evolution of the mode functions is given by:

$$f_k' = \mathcal{F} f_k - g_k, \qquad g_k' = -\mathcal{F} g_k - 4\pi\, \sigma_c\, g_k + k^2\, f_k, \tag{3.14}$$

and the possibility of conducting initial conditions has been included for comparison. In the absence of sources, as already mentioned after Eqs. (3.1)–(3.3), Eqs. (3.14) are invariant under generalized duality transformations stipulating that $f_k \to g_k/k$, $g_k \to -k\, f_k$ and $\mathcal{F} \to -\mathcal{F}$.

In terms of the mode functions, the Fourier components of $\hat{B}_i(\vec{x}, \tau)$ and $\hat{E}_i(\vec{x}, \tau)$ are respectively

$$\hat{B}_i(\vec{q}, \tau) = -i\epsilon_{mni} \sum_{\alpha} e_n^{\alpha}\, q_m [\hat{a}_{\vec{q},\alpha}\; f_q(\tau) + \hat{a}^{\dagger}_{-\vec{q},\alpha}\; f_q^{*}(\tau)], \tag{3.15}$$

$$\hat{E}_i(\vec{q}, \tau) = \sum_{\beta} e_i^{\alpha} [\hat{a}_{\vec{q},\beta}\; g_q(\tau) + \hat{a}^{\dagger}_{-\vec{q},\beta}\; g_q^{*}(\tau)]. \tag{3.16}$$

[6]Note that $e_i^{(\alpha)}(\hat{k})$ (with $\alpha = 1,\, 2$) are two mutually orthogonal unit vectors which are also orthogonal to $\hat{k}$; furthermore $\sum_{\alpha} e_i^{(\alpha)}(\hat{k})\, e_i^{(\alpha)}(\hat{k}) = P_{ij}(\hat{k})$.

It can be immediately checked that Eqs. (3.15)–(3.16) obey the stochastic averages defined earlier in Eqs. (3.5)–(3.6); for instance, in the case of the magnetic field operator,

$$\langle 0|\hat{B}_i(\vec{q},\tau)\hat{B}_j(\vec{p},\tau)|0\rangle = \frac{2\pi^2}{q^3}\,P_{\rm B}(q,\tau)\,P_{ij}(\hat{q})\delta^{(3)}(\vec{q}+\vec{p}), \qquad P_{\rm B}(q,\tau) = \frac{q^5}{2\pi^2}|f_q(\tau)|^2, \quad (3.17)$$

in full analogy with Eq. (3.5). It can be easily argued that Eqs. (3.6) and (3.8)–(3.10) are similarly satisfied with

$$P_{\rm E}(q,\tau) = \frac{q^3}{2\pi^2}|g_q(\tau)|^2, \qquad P_{\rm EB}(q,\tau) = \frac{q^4}{2\pi^2}[f_q^{*}(\tau)g_q(\tau) + f_q(\tau)g_q^{*}(\tau)], \tag{3.18}$$

where $P_{\rm E}(q,\tau)$ denotes the power spectrum of the electric fields and $P_{\rm EB}(q,\tau)$ is the spectrum of the cross-correlation between electric and magnetic fields. To have compatibility between the evolution equations of the power spectra (i.e. Eqs. (3.8)–(3.10)) and the evolution equations of the mode functions (i.e. Eq. (3.14)) the cross-correlation spectrum is essential. Using the power spectra defined earlier the average magnetic and electric energy densities are

$$\overline{\rho}_{\rm B}(\tau) = \frac{1}{4\pi a^4}\int\frac{dq}{q}P_{\rm B}(q,\tau), \qquad \overline{\rho}_{\rm E}(\tau) = \frac{1}{4\pi a^4}\int\frac{dq}{q}P_{\rm E}(q,\tau), \tag{3.19}$$

Backreaction problems are avoided if $\overline{\rho}_{\rm B}$ and $\overline{\rho}_{\rm E}$ are smaller than the background energy density $3H^2\overline{M}_{\rm P}^2$. Moreover the contribution of the electric and magnetic fields to the evolution equations of φ and σ must be subleading. These requirements are, however, less severe than the ones stemming from the predominance of the adiabatic mode discussed in Sec. 6.

3.3 Fluctuations of the energy-momentum tensor

The normalized fluctuation of the energy density are given by

$$\delta\rho_{\rm B}(\vec{x},\tau) = \int\frac{d^3q}{(2\pi)^{3/2}}\,\delta\rho_{\rm B}(\vec{q},\tau)\,e^{-i\vec{q}\cdot\vec{x}}, \qquad \delta\rho_{\rm E}(\vec{x},\tau) = \int\frac{d^3q}{(2\pi)^{3/2}}\,\delta\rho_{\rm E}(\vec{q},\tau)\,e^{-i\vec{q}\cdot\vec{x}}, \tag{3.20}$$

where

$$\begin{aligned}
\delta\rho_{\rm B}(\vec{q},\tau) &= \frac{1}{(2\pi)^{3/2}}\int d^3k\left[B_i(\vec{k},\tau)B_i(\vec{q}-\vec{k},\tau) - \frac{4\pi^2}{k^3}\,P_{\rm B}(k,\tau)\delta^{(3)}(\vec{q})\right],\\
\delta\rho_{\rm E}(\vec{q},\tau) &= \frac{1}{(2\pi)^{3/2}}\int d^3k\left[E_i(\vec{k},\tau)E_i(\vec{q}-\vec{k},\tau) - \frac{4\pi^2}{k^3}\,P_{\rm E}(k,\tau)\delta^{(3)}(\vec{q})\right].
\end{aligned} \tag{3.21}$$

Similarly, the normalized fluctuations of the electric and magnetic pressures are $\delta p_{\rm B}(\vec{x},\tau) = \delta\rho_{\rm B}(\vec{x},\tau)/3$ and $\delta p_{\rm E}(\vec{x},\tau) = \delta\rho_{\rm E}(\vec{x},\tau)/3$. The electric and magnetic anisotropic stresses are

$$\Pi_{ij}^{(\rm B)}(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3q\Pi_{ij}^{(B)}(\vec{q},\tau)\,e^{-i\vec{q}\cdot\vec{x}}, \quad \Pi_{ij}^{(\rm E)}(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3q\Pi_{ij}^{(E)}(\vec{q},\tau)\,e^{-i\vec{q}\cdot\vec{x}}, \tag{3.22}$$

where

$$\Pi_{ij}^{(B)}(\vec{q},\tau) = \frac{1}{4\pi a^4}\int \frac{d^3k}{(2\pi)^{3/2}}\biggl[B_i(\vec{k},\tau)B_j(\vec{q}-\vec{k},\tau) - \frac{\delta_{ij}}{3}B_m(\vec{k},\tau)B_m(\vec{q}-\vec{k},\tau)\biggr], \tag{3.23}$$

$$\Pi_{ij}^{(E)}(\vec{q},\tau) = \frac{1}{4\pi a^4}\int \frac{d^3k}{(2\pi)^{3/2}}\biggl[E_i(\vec{k},\tau)E_j(\vec{q}-\vec{k},\tau) - \frac{\delta_{ij}}{3}E_m(\vec{k},\tau)E_m(\vec{q}-\vec{k},\tau)\biggr]. \tag{3.24}$$

It is practical to introduce the scalar projections of the electric and magnetic anisotropic stresses

$$\nabla^2\Pi_{\mathrm{B}}(\vec{x},\tau) = \partial_i\partial_j\Pi_{\mathrm{B}}^{ij}(\vec{x},\tau), \qquad \nabla^2\Pi_{\mathrm{E}}(\vec{x},\tau) = \partial_i\partial_j\Pi_{\mathrm{E}}^{ij}(\vec{x},\tau), \tag{3.25}$$

entering the evolution equations of the scalar modes of the geometry. Note that the stochastic averages of the fluctuations variables defined in Eqs. (3.20)–(3.21) and (3.22)–(3.24) are all vanishing, i.e. using Eqs. (3.5)–(3.6), $\langle\delta\rho_{\mathrm{B}}(\vec{x},\tau)\rangle = 0$ and $\langle\delta\rho_{\mathrm{E}}(\vec{x},\tau)\rangle = 0$. The second order correlations of the energy density fluctuations and of the anisotropic stresses are defined as

$$\langle\delta\rho_X(\vec{q},\tau)\,\delta\rho_X(\vec{p},\tau)\rangle = \frac{2\pi^2}{q^3}\mathcal{Q}_X(q,\tau)\,\delta^{(3)}(\vec{q}+\vec{p}), \tag{3.26}$$

$$\langle\Pi_X(\vec{q},\tau)\,\Pi_X(\vec{p},\tau)\rangle = \frac{2\pi^2}{q^3}\mathcal{Q}_{X\Pi}(q,\tau)\,\delta^{(3)}(\vec{q}+\vec{p}), \tag{3.27}$$

where $X = \mathrm{B},\, E$ leading, overall, to four independent spectra

$$\mathcal{Q}_{\mathrm{B}}(q,\tau) = \frac{q^3}{128\,\pi^3\,a^8}\int d^3k\frac{P_{\mathrm{B}}(k,\tau)}{k^3}\frac{P_{\mathrm{B}}(|\vec{q}-\vec{k}|,\tau)}{|\vec{q}-\vec{k}|^3}\,\Lambda_\rho(k,q), \tag{3.28}$$

$$\mathcal{Q}_{\mathrm{E}}(q,\tau) = \frac{q^3}{128\,\pi^3\,a^8}\int d^3k\frac{P_{\mathrm{E}}(k,\tau)}{k^3}\frac{P_{\mathrm{E}}(|\vec{q}-\vec{k}|,\tau)}{|\vec{q}-\vec{k}|^3}\,\Lambda_\rho(k,q), \tag{3.29}$$

$$\mathcal{Q}_{\mathrm{B}\Pi}(q,\tau) = \frac{q^3}{288\,\pi^3\,a^8(\tau)}\int d^3k\frac{P_{\mathrm{B}}(k,\tau)}{k^3}\frac{P_{\mathrm{B}}(|\vec{q}-\vec{k}|,\tau)}{|\vec{q}-\vec{k}|^3}\,\Lambda_\Pi(k,q), \tag{3.30}$$

$$\mathcal{Q}_{\mathrm{E}\Pi}(q,\tau) = \frac{q^3}{288\,\pi^3\,a^8(\tau)}\int d^3k\frac{P_{\mathrm{E}}(k,\tau)}{k^3}\frac{P_{\mathrm{E}}(|\vec{q}-\vec{k}|,\tau)}{|\vec{q}-\vec{k}|^3}\,\Lambda_\Pi(k,q). \tag{3.31}$$

The functions $\Lambda_\rho(k,q)$ and $\Lambda_\Pi(k,q)$ are defined as

$$\Lambda_\rho(k,q) = 1 + \frac{[\vec{k}\cdot(\vec{q}-\vec{k})]^2}{k^2|\vec{q}-\vec{k}|^2}, \tag{3.32}$$

$$\begin{aligned}\Lambda_\Pi(k,q) &= 1 + \frac{[\vec{k}\cdot(\vec{q}-\vec{k})]^2}{k^2|\vec{q}-\vec{k}|^2} + \frac{6}{q^2}\biggl[\vec{k}\cdot(\vec{q}-\vec{k}) - \frac{[\vec{k}\cdot(\vec{q}-\vec{k})]^3}{k^2|\vec{q}-\vec{k}|^2}\biggr] \\ &+ \frac{9}{q^4}\biggl[k^2|\vec{q}-\vec{k}|^2 - 2[\vec{k}\cdot(\vec{q}-\vec{k})]]^2 + \frac{[\vec{k}\cdot(\vec{q}-\vec{k})]^4}{k^2|\vec{q}-\vec{k}|^2}\biggr].\end{aligned} \tag{3.33}$$

The functions $\Lambda_\rho(k,q)$ and $\Lambda_\Pi(k,q)$ coincide for magnetic and electric degrees of freedom since both $\vec{E}$ and $\vec{B}$ are solenoidal fields: $\vec{B}$ is solenoidal because of the absence of magnetic

monopoles while $\vec{E}$ is solenoidal because the pprotoinflationary plasma is globally neutral and any electric charge asymmetry is absent. The explicit expressions of the power spectra of Eqs. (3.28)–(3.31) are presented and discussed in appendix B in the case of a monotonic growth rate.

3.4 Eenergy-momentum tensor evolution

Instead of computing $\delta\rho_{\rm B}$, $\delta\rho_{\rm E}$ and the other relevant components from the field variables, it is practical to deduce the evolution equations obeyed by these quantities. The various components of the energy-momentum tensor can be written, in real space, as

$$\mathcal{Z}_0^0 = \overline{\rho}_{\rm B} + \overline{\rho}_{\rm E} + \delta\rho_{\rm B} + \delta\rho_{\rm E}, \tag{3.34}$$

$$\mathcal{Z}_i^0 = -\frac{(\vec{E}\times\vec{B})^i}{4\pi a^4}, \qquad \delta\mathcal{Z}_0^i = \frac{(\vec{E}\times\vec{B})^i}{4\pi a^4}, \tag{3.35}$$

$$\mathcal{Z}_i^j = -(\overline{p}_{\rm E} + \overline{p}_{\rm B})\delta_i^j - (\delta p_{\rm E} + \delta p_{\rm B})\delta_i^j + \Pi_i^{({\rm E})\,j} + \Pi_i^{({\rm B})\,j}. \tag{3.36}$$

Using Eqs. (3.34)–(3.36) and the explicit expression of the covariant derivative, Eq. (1.6) demands, in components,

$$\partial_\tau(\delta\rho_{\rm E} + \delta\rho_{\rm B}) + 4\mathcal{H}(\delta\rho_{\rm E} + \delta\rho_{\rm B}) = 2\mathcal{F}(\delta\rho_{\rm B} - \delta\rho_{\rm E}) - P - \frac{\vec{J}\cdot\vec{E}}{a^4}, \tag{3.37}$$

$$\partial_\tau P + 4\mathcal{H}P = -\frac{\vec{\nabla}\cdot(\vec{J}\times\vec{B})}{a^4} - \nabla^2[\delta p_{\rm B} + \delta p_{\rm E} - (\Pi_{\rm B} + \Pi_{\rm E})], \tag{3.38}$$

where $P = \vec{\nabla}\cdot\vec{S}$ denotes the three-divergence of the Poynting vector $\vec{S} = (\vec{E}\times\vec{B})/(4\pi a^4)$. In Eq. (3.38) the terms $\partial_i[(\partial^i\lambda)/\lambda](\delta\rho_{\rm B} - \delta\rho_{\rm E})$ and $\partial_i\lambda\partial^i(\delta\rho_{\rm B} - \delta\rho_{\rm E})/\lambda$ have been neglected since they couple spatial gradients of the growth rate and magnetic inhomogeneities. These terms are of higher order in the present description. Furthermore, using standard vector identities[7] Eq. (3.38) can be recast in the following form:

$$\partial_\tau P + 4\mathcal{H}P = -\frac{\vec{\nabla}\cdot(\vec{J}\times\vec{B})}{a^4} + \frac{\vec{\nabla}\cdot[(\vec{\nabla}\times\vec{B})\times\vec{B}] + \vec{\nabla}\cdot[(\vec{\nabla}\times\vec{E})\times\vec{E}]}{4\pi a^4}. \tag{3.39}$$

The evolution of the difference between $\delta\rho_{\rm B}$ and $\delta\rho_{\rm E}$ can be obtained directly from Eqs. (3.1)–(3.3):

$$\partial_\tau(\delta\rho_{\rm B} - \delta\rho_{\rm E}) + 4\mathcal{H}(\delta\rho_{\rm B} - \delta\rho_{\rm E}) = 2\mathcal{F}(\delta\rho_{\rm E} + \delta\rho_{\rm B}) - \frac{\vec{B}\cdot\vec{\nabla}\times\vec{E} + \vec{E}\cdot\vec{\nabla}\times\vec{B}}{4\pi a^4} + \frac{\vec{E}\cdot\vec{J}}{a^4}. \tag{3.40}$$

The system of Eqs. (3.37)–(3.40) can be studied in various approximations (subleading spatial gradients, large conductivity limit and so on and so forth). In the most naive case $P(\vec{x},\tau)$ simply scales as a^{-4}. This can be easily understood since, up to spatial gradients, the evolution of P does not depend on the growth rate. Conversely, the time derivative of P is proportional to the Laplacians of the pressures and of the anisotropic stresses.

[7]Given a solenoidal vector field C_i, (such as $\vec{B}$ or $\vec{E}$) the product $\partial_i C_j \partial^j C^i$ can be expressed as $\vec{\nabla}\cdot[(\vec{\nabla}\times\vec{C})\times\vec{C}] + \nabla^2 C^2/2$.

4 Quasi-normal modes

The evolution of the scalar modes of the geometry, of the inflaton and of the spectator field are all coupled to the scalar inhomogeneities of the gauge sector. This system will now be reduced to the evolution of its quasi-normal modes whose equations are coupled but, most importantly, decoupled from all the other perturbation variables. The considerations of the present section and of the appendix A can be easily generalized to various situations involving, for instance, more than one spectator field.

4.1 Uniform curvature hypersurfaces

The scalar fluctuations of the four-dimensional metric are parametrized by four different functions whose number can be eventually reduced by specifying (either completely or partially) the coordinate system:

$$\delta_{\rm s} g_{00} = 2a^2\phi, \qquad \delta_{\rm s} g_{ij} = 2a^2(\psi\delta_{ij} - \partial_i\partial_j\alpha), \qquad \delta_{\rm s} g_{0i} = -a^2\partial_i\beta, \tag{4.1}$$

where $\delta_{\rm s}$ denotes the scalar mode of the corresponding tensor component; the full metric (i.e. background plus inhomogeneities) is given, in these notations, by $g_{\alpha\beta}(\vec{x},\tau) = \overline{g}_{\alpha\beta}(\tau) + \delta_{\rm s} g_{\alpha\beta}(\vec{x},\tau)$ where, as already mentioned prior to Eqs. (2.14)–(2.17) $\overline{g}_{\alpha\beta}(\tau) = a^2(\tau)\eta_{\alpha\beta}$. For infinitesimal coordinate shifts $\tau \to \overline{\tau} = \tau + \epsilon_0$ and $x^i \to \overline{x}^i = x^i + \partial^i\epsilon$ the functions $\phi(\vec{x},\tau)$, $\beta(\vec{x},\tau)$, $\psi(\vec{x},\tau)$ and $\alpha(\vec{x},\tau)$ introduced in Eq. (4.1) transform as[8] :

$$\phi \to \overline{\phi} = \phi - \mathcal{H}\epsilon_0 - \epsilon_0', \qquad \psi \to \overline{\psi} = \psi + \mathcal{H}\epsilon_0, \tag{4.2}$$

$$\beta \to \overline{\beta} = \beta + \epsilon_0 - \epsilon', \qquad \alpha \to \overline{\alpha} = \alpha - \epsilon. \tag{4.3}$$

In the uniform curvature gauge two out of the four functions of Eq. (4.1) are set to zero [8, 9, 10]:

$$\alpha = 0, \qquad \psi = 0, \qquad \phi = \phi(\vec{x},\tau), \qquad \beta = \beta(\vec{x},\tau). \tag{4.4}$$

Starting from a gauge where α and ψ do not vanish, the perturbed line element can always be brought in the form (4.4) by demanding $\overline{\alpha} = 0$ and $\overline{\psi} = 0$ in Eqs. (4.2) and (4.3). If $\alpha \neq 0$ and $\psi \neq 0$, the uniform curvature gauge condition can be recovered by fixing the gauge parameters as $\epsilon = \alpha$ and $\epsilon_0 = -\psi/\mathcal{H}$. This choice guarantees that, in the transformed coordinate system, $\overline{\psi} = \overline{\alpha} = 0$.

A convenient gauge choice is essential for a sound treatment of problems involving the presence of anisotropic stresses. The conformally Newtonian gauge is known to be unsuitable for the analysis of perturbative systems where the anisotropic stresses play an important role. Similar caveats arise in the discussion of the Einstein-Boltzmann hierarchy whenever

[8]The slow roll parameter ϵ must not be confused with the parameter of the gauge transformation. These two variables never appear together either in the preceding or in the following discussion so that no confusion is possible.

the entropic initial conditions are dominated by the anisotropic stresses as it happens in the neutrino sector. Both points have been addressed long ago when discussing the initial conditions for the magnetized CMB anisotropies [28] (see also [12] and references therein). Other gauges are also suitable for the treatment of magnetized inhomogeneities but we shall not discuss them here. As already mentioned in Sec. 1, to avoid lengthy digressions the full set of evolution equations has been presented and discussed in the appendix A.

4.2 The decoupled system

Consider, to begin with, the evolution equations for the fluctuations of the inflaton (i.e. χ_{φ}) and of the spectator field (i.e. χ_{σ}) which are reported in Eqs. (A.20) and (A.21). Recalling that $\Delta_{\sigma} = \nabla^2 \chi_{\sigma}$ and $\Delta_{\varphi} = \nabla^2 \chi_{\varphi}$, from the momentum constraint of Eq. (A.9) (neglecting a generic fluid contribution which is anyway irrelevant during the inflationary phase) the following relation holds:

$$\Delta_{\phi} = 4\pi G\biggl[\biggl(\frac{\varphi'}{\mathcal{H}}\biggr)\Delta_{\varphi} + \biggl(\frac{\sigma'}{\mathcal{H}}\biggr)\Delta_{\sigma}\biggr] - \frac{4\pi G a^2}{\mathcal{H}} P. \tag{4.5}$$

During inflation the three-divergence of the Poynting vector P decreases always as a^{-4} so the predominant contribution to the curvature perturbations on uniform curvature hypersurfaces is given by the first two terms of Eq. (4.5). There is, however, an important proviso: the time derivative of Δ_{ϕ} (i.e. Δ_{ϕ}') appearing in Eqs. (A.20) and (A.21) leads to a term going as $P' = \partial_{\tau} P$ containing the Laplacians of the magnetic and electric energy density fluctuations (see Eq. (3.38)). It is advisable, as usual, to assess the relative weight of different terms not at the beginning, but rather at the end of the derivation.

In the gauge (4.4), the curvature perturbations on comoving orthogonal hypersurfaces, customarily denoted by $\mathcal{R}$, coincides with ϕ up to a background dependent coefficients,

$$\mathcal{R} = -\frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}'}\,\phi. \tag{4.6}$$

Defining $\Delta_{\mathcal{R}} = \nabla^2 \mathcal{R}$ and recalling Eq. (4.6), we have, from Eq. (4.5),

$$\Delta_{\mathcal{R}} = -\biggl[\frac{\mathcal{H}\varphi'}{\varphi'^2 + \sigma'^2}\Delta_{\varphi} + \frac{\mathcal{H}\sigma'}{\varphi'^2 + \sigma'^2}\Delta_{\sigma}\biggr] + \frac{\mathcal{H}a^2}{\varphi'^2 + \sigma'^2} P. \tag{4.7}$$

The equations describing the dynamics of the quasi-normal modes is obtained by eliminating Δ_{ϕ}, Δ_{ϕ}' and $\nabla^2\Delta_{\beta}$ from Eqs. (A.20) and (A.21). Equation (4.5) gives Δ_{ϕ} in terms of Δ_{φ}, Δ_{σ} and P. The combination $(\Delta_{\phi}' + \nabla^2 \Delta_{\beta})$ can then be obtained, after some algebra, from the explicit expression of Δ_{ϕ}' and from the Hamiltonian constraint of Eq. (A.18) (see also Eq. (A.8)).

Thus, as discussed in appendix A, Eqs. (A.23) and (A.25) can be inserted into Eqs. (A.20)–(A.21) and the resulting system becomes:

$$\Delta_{\varphi}'' + 2\mathcal{H}\Delta_{\varphi}' - \nabla^2\Delta_{\varphi} + \mathcal{A}_{\varphi\varphi}\Delta_{\varphi} + \mathcal{A}_{\varphi\sigma}\Delta_{\sigma} + \mathcal{S}_{\varphi} = 0, \tag{4.8}$$

$$\Delta_{\sigma}'' + 2\mathcal{H}\Delta_{\sigma}' - \nabla^2\Delta_{\sigma} + \mathcal{A}_{\sigma\sigma}\Delta_{\sigma} + \mathcal{A}_{\sigma\varphi}\Delta_{\varphi} + \mathcal{S}_{\sigma} = 0. \tag{4.9}$$

The coefficients $\mathcal{A}_{\varphi\varphi}$, $\mathcal{A}_{\sigma\sigma}$ and $\mathcal{A}_{\varphi\sigma} = \mathcal{A}_{\sigma\varphi}$ depend on the background and are:

$$\mathcal{A}_{\varphi\varphi} = a^2 \frac{\partial^2 V}{\partial\varphi^2} + \frac{1}{\overline{M}_{\mathrm{P}}^2}\biggl[2a^2 \frac{\partial V}{\partial\varphi}\biggl(\frac{\varphi'}{\mathcal{H}}\biggr) + \biggl(2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\biggr)\varphi'^2\biggr], \tag{4.10}$$

$$\mathcal{A}_{\varphi\sigma} = \mathcal{A}_{\sigma\varphi} = \frac{1}{\overline{M}_{\mathrm{P}}^2}\biggl[a^2 \frac{\partial V}{\partial\varphi}\biggl(\frac{\sigma'}{\mathcal{H}}\biggr) + \frac{\partial W}{\partial\sigma}(\frac{\varphi'}{\mathcal{H}}\biggr) + \biggl(2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\biggr)\varphi'\,\sigma'\biggr], \tag{4.11}$$

$$\mathcal{A}_{\sigma\sigma} = a^2 \frac{\partial^2 W}{\partial\sigma^2} + \frac{1}{\overline{M}_{\mathrm{P}}^2}\biggl[2a^2 \frac{\partial W}{\partial\sigma}\biggl(\frac{\sigma'}{\mathcal{H}}\biggr) + \biggl(2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\biggr)\sigma'^2\biggr], \tag{4.12}$$

where, comparing with the expressions of the appendix A, the four-dimensional Palnck mass defined in Eq. (2.5) has been introduced by trading $8\pi G$ for $1/\overline{M}_{\mathrm{P}}^2$. The source terms $\mathcal{S}_\varphi$ and $\mathcal{S}_\sigma$ appearing in Eqs. (4.8) and (4.9) are:

$$\mathcal{S}_\varphi = \frac{a^2}{2\overline{M}_{\mathrm{P}}^2}\biggl(\frac{\varphi'}{\mathcal{H}}\biggr)\biggl[P' - 2\biggl(\frac{\mathcal{H}'}{\mathcal{H}} + \frac{a^2}{\varphi'}\frac{\partial V}{\partial\varphi}\biggr)P + \nabla^2(\delta\rho_{\mathrm{B}} + \delta\rho_{\mathrm{E}})\biggr] + \frac{a^2}{\lambda}\frac{\partial\lambda}{\partial\varphi}\nabla^2(\delta\rho_{\mathrm{B}} - \delta\rho_{\mathrm{E}}),$$

$$\mathcal{S}_\sigma = \frac{a^2}{2\overline{M}_{\mathrm{P}}^2}\biggl(\frac{\sigma'}{\mathcal{H}}\biggr)\biggl[P' - 2\biggl(\frac{\mathcal{H}'}{\mathcal{H}} + \frac{a^2}{\sigma'}\frac{\partial W}{\partial\sigma}\biggr)P + \nabla^2(\delta\rho_{\mathrm{B}} + \delta\rho_{\mathrm{E}})\biggr] + \frac{a^2}{\lambda}\frac{\partial\lambda}{\partial\sigma}\nabla^2(\delta\rho_{\mathrm{B}} - \delta\rho_{\mathrm{E}}),$$

which are expressible in a slightly different form by using the evolution equations of φ, σ together with the governing equation for P, i.e., respectively, Eqs. (2.16), (2.17) and (3.38). The result of this manipulation, neglecting the spatial gradients of λ is:

$$\mathcal{S}_\varphi = \frac{a^2}{2\overline{M}_{\mathrm{P}}^2}\biggl(\frac{\varphi'}{\mathcal{H}}\biggr)\biggl[2\biggl(\frac{\varphi'}{\mathcal{H}}\biggr)'\biggl(\frac{\mathcal{H}}{\varphi'}\biggr)P + \mathcal{V}_{\mathrm{EB}}\biggr] + \frac{a^2}{\lambda}\frac{\partial\lambda}{\partial\varphi}\nabla^2(\delta\rho_{\mathrm{B}} - \delta\rho_{\mathrm{E}}), \tag{4.13}$$

$$\mathcal{S}_\sigma = \frac{a^2}{2\overline{M}_{\mathrm{P}}^2}\biggl(\frac{\sigma'}{\mathcal{H}}\biggr)\biggl[2\biggl(\frac{\sigma'}{\mathcal{H}}\biggr)'\biggl(\frac{\mathcal{H}}{\sigma'}\biggr)P + \mathcal{V}_{\mathrm{EB}}\biggr] + \frac{a^2}{\lambda}\frac{\partial\lambda}{\partial\sigma}\nabla^2(\delta\rho_{\mathrm{B}} - \delta\rho_{\mathrm{E}}), \tag{4.14}$$

where $\mathcal{V}_{\mathrm{EB}}$ is defined as

$$\mathcal{V}_{\mathrm{EB}} = \frac{2}{3}\nabla^2(\delta\rho_{\mathrm{B}} + \delta\rho_{\mathrm{E}}) + \nabla^2(\Pi_{\mathrm{B}} + \Pi_{\mathrm{E}}) - \frac{\vec{\nabla}\cdot(\vec{J}\times\vec{B})}{a^4}. \tag{4.15}$$

The system of equations derived here is the starting point for the determination of the power spectra of curvature perturbations to be analyzed in the forthcoming sections.

5 Magnetized power spectra of the scalar modes

5.1 Simplifying approximations

The solution of Eqs. (4.8) and (4.9) during a phase of slow roll expansion determines the large-scale power spectra of curvature perturbations. The expressions of the coefficients of Eqs. (4.10)–(4.12) can be simplified in the limits $\rho_\sigma \ll \rho_\varphi$ and $\epsilon \simeq \eta \ll 1$ and it can be shown, for instance, that

$$\mathcal{A}_{\varphi\varphi} = \frac{z_\varphi''}{z_\varphi} + \frac{a''}{a} \gg \mathcal{A}_{\varphi\sigma}, \qquad z_\varphi = \frac{a\varphi'}{\mathcal{H}}. \tag{5.1}$$

The second inequality appearing in Eq. (5.1) is derived by appreciating that $\mathcal{A}_{\varphi\sigma}$ can be recast in the following form:

$$\mathcal{A}_{\varphi\sigma} = \frac{H^2 a^2}{2\overline{M}_{\mathrm{P}}^2}\biggl[\Bigl(\frac{\dot{\sigma}}{H}\Bigr)\Bigl(\frac{V_{,\varphi}}{H^2}\Bigr) + \Bigl(\frac{\dot{\varphi}}{H}\Bigr)\Bigl(\frac{W_{,\sigma}}{H^2}\Bigr) + \Bigl(3 + \frac{\dot{H}}{H^2}\Bigr)\Bigl(\frac{\dot{\varphi}}{H}\Bigr)\Bigl(\frac{\dot{\sigma}}{H}\Bigr)\biggr]. \tag{5.2}$$

and by using known identities of the slow roll dynamics[9] For the simlars reasons $\mathcal{A}_{\sigma\sigma}$ must be smaller than $(\mathcal{H}^2 + \mathcal{H}')$. Neglecting the subleading terms in Eqs. (4.13)–(4.14) $\mathcal{S}_{\varphi}$ and $\mathcal{S}_{\sigma}$ become

$$\mathcal{S}_{\varphi} = \frac{a^2}{\overline{M}_{\mathrm{P}}}\,\nabla^2\overline{\mathcal{S}}_{\varphi}, \qquad \mathcal{S}_{\sigma} = \frac{a^2}{\overline{M}_{\mathrm{P}}}\,\nabla^2\overline{\mathcal{S}}_{\sigma}, \tag{5.3}$$

where the two dimensionless variables $\overline{\mathcal{S}}_{\varphi}$ and $\overline{\mathcal{S}}_{\sigma}$ have been introduced:

$$\overline{\mathcal{S}}_{\varphi} = \biggl[\Bigl(\frac{\varphi'}{3\overline{M}_{\mathrm{P}}\,\mathcal{H}} + \frac{\overline{M}_{\mathrm{P}}}{\lambda}\frac{\partial\lambda}{\partial\varphi}\Bigr)\delta\rho_{\mathrm{B}} + \Bigl(\frac{\varphi'}{3\overline{M}_{\mathrm{P}}\,\mathcal{H}} - \frac{\overline{M}_{\mathrm{P}}}{\lambda}\frac{\partial\lambda}{\partial\varphi}\Bigr)\delta\rho_{\mathrm{E}}\biggr] + \frac{\varphi'(\Pi_{\mathrm{B}} + \Pi_{\mathrm{E}})}{2\overline{M}_{\mathrm{P}}\,\mathcal{H}}, \tag{5.4}$$

$$\overline{\mathcal{S}}_{\sigma} = \biggl[\Bigl(\frac{\sigma'}{3\overline{M}_{\mathrm{P}}\,\mathcal{H}} + \frac{\overline{M}_{\mathrm{P}}}{\lambda}\frac{\partial\lambda}{\partial\sigma}\Bigr)\delta\rho_{\mathrm{B}} + \Bigl(\frac{\sigma'}{3\overline{M}_{\mathrm{P}}\,\mathcal{H}} - \frac{\overline{M}_{\mathrm{P}}}{\lambda}\frac{\partial\lambda}{\partial\sigma}\Bigr)\delta\rho_{\mathrm{E}}\biggr] + \frac{\sigma'(\Pi_{\mathrm{B}} + \Pi_{\mathrm{E}})}{2\overline{M}_{\mathrm{P}}\,\mathcal{H}}. \tag{5.5}$$

The terms containing the Ohmic current shall be neglected and the whole effect of conducting initial conditions will be simply encoded in the further suppression of the electric components as explained in Appendix B.

Introducing then the rescaled variables $q_{\varphi} = a\chi_{\varphi}$ and $q_{\sigma} = a\chi_{\sigma}$ and recalling that $\Delta_{\varphi} = \nabla^2\chi_{\varphi}$ and $\Delta_{\sigma} = \nabla^2\chi_{\sigma}$, the Laplacians can be eliminated from the left and from the right-hand sides of Eqs. (4.8)–(4.9) so that the resulting equations assume the following simplified form[10]:

$$q_{\varphi}'' - \nabla^2 q_{\varphi} - \frac{z_{\varphi}''}{z_{\varphi}} q_{\varphi} + \frac{a^3}{\overline{M}_{\mathrm{P}}}\,\overline{S}_{\varphi}(\vec{x},\tau) = 0, \tag{5.6}$$

$$q_{\sigma}'' - \nabla^2 q_{\sigma} - \frac{a''}{a} q_{\sigma} + \frac{a^3}{\overline{M}_{\mathrm{P}}}\,\overline{S}_{\sigma}(\vec{x},\tau) = 0. \tag{5.7}$$

In the class of models introduced in Eqs. (2.21)–(2.23) the slow roll parameters are given by $\epsilon = \eta = 1/\alpha$ with $\alpha \gg 1$ and $z_{\varphi} \propto a(\tau)$. Thus the coefficients $\mathcal{A}_{\varphi\sigma}$ and $\mathcal{A}_{\sigma\sigma}$ become:

$$\mathcal{A}_{\varphi\sigma} = -\frac{2\sqrt{2}(3\alpha - 1)}{\tau^2\sqrt{\alpha}\,(1-\alpha)^2}\Bigl(\frac{M}{\overline{M}_{\mathrm{P}}^2}\Bigr)^2, \quad \mathcal{A}_{\sigma\sigma} = \frac{2(3\alpha - 1)}{(1-\alpha)^2\,\tau^2}\biggl[1 - \frac{2}{\alpha}\Bigl(\frac{M}{\overline{M}_{\mathrm{P}}^2}\Bigr)^2\biggr]. \tag{5.8}$$

which are both suppressed in the limit $\epsilon \ll 1$ (i.e. $\alpha \gg 1$). The coefficient $\mathcal{A}_{\varphi\sigma}$ is further suppressed because $M \ll \overline{M}_{\mathrm{P}}$, as implied by the subdominant nature of σ. This example illustrates concretely the nature of the general approximations analyzed in this section.

[9] In particular recall that $(V_{,\varphi}/H^2) = 3\sqrt{2}\,\sqrt{\epsilon}\,\overline{M}_{\mathrm{P}}$ and that $\dot{\varphi}/H = \sqrt{2}\,\overline{M}_{\mathrm{P}}\,\sqrt{\epsilon}$; furthermore the subdominance of σ stipulates that $\dot{\sigma} \ll H\,\overline{M}_{\mathrm{P}}$.

[10] Note that the pump field of Eq. (5.7) is not given by z_{σ}''/z_{σ} (with $z_{\sigma} = a\sigma'/\mathcal{H}$). This lack of symmetry is ultimately related to the subdominant nature of σ.

5.2 Primary and secondary power spectra

In Fourier space Eqs. (5.7) and (5.8) become:

$$q_{\varphi}'' + \left[k^2 - \frac{z_{\varphi}''}{z_{\varphi}}\right]q_{\varphi} = -\frac{a^3}{\overline{M}_{\mathrm{P}}}\,\overline{S}_{\varphi}(\vec{k},\tau), \tag{5.9}$$

$$q_{\sigma}'' + \left[k^2 - \frac{a''}{a}\right]q_{\sigma} = -\frac{a^3}{\overline{M}_{\mathrm{P}}}\,\overline{S}_{\sigma}(\vec{k},\tau), \tag{5.10}$$

and their corresponding solutions are

$$q_{\varphi}(\vec{k},\tau) = q_{\varphi}^{(1)}(\vec{k},\tau) - \frac{1}{\overline{M}_{\mathrm{P}}}\int_{\tau_*}^{\tau} a^3(\tau')\,\overline{S}_{\varphi}(k,\tau')\,G_k^{(\varphi)}(\tau',\tau)\,d\tau', \tag{5.11}$$

$$q_{\sigma}(\vec{k},\tau) = q_{\sigma}^{(1)}(\vec{k},\tau) - \frac{1}{\overline{M}_{\mathrm{P}}}\int_{\tau_*}^{\tau} a^3(\tau')\,\overline{S}_{\sigma}(k,\tau')\,G_k^{(\sigma)}(\tau',\tau)\,d\tau', \tag{5.12}$$

where $G_k^{(\varphi)}(\tau',\tau)$ and $G_k^{(\sigma)}(\tau',\tau)$ denote the Green's function obtained from the appropriately normalized mode functions of the corresponding homogeneous equations. Denoting with $F(k,\tau)$ and $F^*(k,\tau)$ the two independent solutions of the homogeneous equation, the corresponding Green's function is:

$$G_k(\tau',\tau) = \frac{F(k,\tau')\,F^*(k,\tau) - F(k,\tau)\,F^*(k,\tau')}{W(\tau')} \tag{5.13}$$

where $W(\tau') = [F'(k\tau')\,F^*(k,\tau') - F^{*\prime}(k,\tau')F(k,\tau')]$ is the Wronskian of the solutions. The explicit form of the mode functions for q_{φ} and q_{σ} are:

$$F_{\varphi}(k,\tau) = \frac{\mathcal{N}_{\varphi}}{\sqrt{2k}}\sqrt{-k\tau}H_{\mu}^{(1)}(-k\tau), \qquad \mu = \frac{3+\epsilon+2\eta}{2(1-\epsilon)}, \tag{5.14}$$

$$F_{\sigma}(k,\tau) = \frac{\mathcal{N}_{\sigma}}{\sqrt{2k}}\sqrt{-k\tau}H_{\tilde{\mu}}^{(1)}(-k\tau), \qquad \tilde{\mu} = \frac{(3-\epsilon)}{2(1-\epsilon)}. \tag{5.15}$$

The expression of the Green's function depends on the indices μ and $\tilde{\mu}$ of the corresponding Hankel functions [35, 36]. Since $\epsilon \ll 1$ and $\eta \ll 1$, the Bessel indices μ and $\tilde{\mu}$ can be expanded in powers of the slow roll parameters and $\mu \simeq 3/2 + 2\epsilon + \eta$ and $\tilde{\mu} = 3/2 + \epsilon$. Consequently, to leading order in the slow roll expansion $\mu \simeq \tilde{\mu} = 3/2$ and this explains why, in this limit, the explicit expressions of $G_k^{(\varphi)}(\tau',\tau)$ and $G_k^{(\sigma)}(\tau',\tau)$ coincide:

$$G_k(\tau',\tau) = \frac{1}{k}\left\{\frac{\tau'-\tau}{k\,\tau'\,\tau}\cos\left[k(\tau'-\tau)\right] - \left(\frac{1}{k^2\tau'\tau}+1\right)\sin\left[k(\tau'-\tau)\right]\right\}. \tag{5.16}$$

Recalling that $q_{\varphi}^{(1)}(\vec{k},\tau)$ and $q_{\sigma}^{(1)}(\vec{k},\tau)$ denote the solutions of the homogeneous equations (5.9) and (5.10) the primary power spectra can be computed from Eqs. (5.14) and (5.15):

$$\langle q_{\varphi}^{(1)}(\vec{k},\tau)q_{\varphi}^{(1)}(\vec{p},\tau)\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\varphi}(k,\tau)\delta^{(3)}(\vec{k}+\vec{p}), \qquad \mathcal{P}_{\varphi}(k,\tau) = \frac{k^3}{2\pi^2}|F_{\varphi}(k,\tau)|^2, \tag{5.17}$$

$$\langle q_{\sigma}^{(1)}(\vec{k},\tau)q_{\sigma}^{(1)}(\vec{p},\tau)\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\sigma}(k,\tau)\delta^{(3)}(\vec{k}+\vec{p}), \qquad \mathcal{P}_{\sigma}(k,\tau) = \frac{k^3}{2\pi^2}|F_{\sigma}(k,\tau)|^2. \tag{5.18}$$

After integration over τ' the final expression can be written as:

$$q_{\varphi}(\vec{k},\tau) = q_{\varphi}^{(1)}(\vec{k},\tau) - \overline{M}_{\rm P}\Big[c_{\varphi}\Omega_{\rm B}(\vec{k},\tau) + d_{\varphi}\Omega_{\rm B\Pi}(\vec{k},\tau)\Big]a(\tau), \tag{5.19}$$

$$q_{\sigma}(\vec{k},\tau) = q_{\sigma}^{(1)}(\vec{k},\tau) - \overline{M}_{\rm P}\Big[c_{\sigma}\Omega_{\rm B}(\vec{k},\tau) + d_{\sigma}\Omega_{\rm B\Pi}(\vec{k},\tau)\Big]a(\tau), \tag{5.20}$$

where the sources have been evaluated to leading order in $k\tau$ and

$$\Omega_{\rm B}(\vec{k},\tau) = \frac{\delta\rho_{\rm B}(\vec{k},\tau)}{3H^2\overline{M}_{\rm P}^2}, \qquad \Omega_{\rm B\Pi}(\vec{k},\tau) = \frac{\Pi_{\rm B}(\vec{k},\tau)}{3H^2\overline{M}_{\rm P}^2}. \tag{5.21}$$

The solutions (5.19)–(5.21) neglects the decreasing electric modes since they are immaterial for the scales that had the longest time to grow and got larger than the Hubble radius between the last 65 and 53 efolds of inflationary expansion (see also beginning of Sec. 6). The rate of decrease of the electric modes is discussed in appendix B for the interested reader. The coefficients appearing in Eqs. (5.19) and (5.20) are slowly varying functions of τ

$$c_{\varphi} = m(f,\epsilon)\left[\frac{1}{3\overline{M}_{\rm P}}\left(\frac{\varphi'}{\mathcal{H}}\right) + \frac{\overline{M}_{\rm P}}{\lambda}\left(\frac{\partial\lambda}{\partial\varphi}\right)\right], \qquad d_{\varphi} = m(f,\epsilon)\,\frac{\overline{M}_{\rm P}}{\lambda}\left(\frac{\partial\lambda}{\partial\varphi}\right), \tag{5.22}$$

$$c_{\sigma} = m(f,\epsilon)\left[\frac{1}{3\overline{M}_{\rm P}}\left(\frac{\sigma'}{\mathcal{H}}\right) + \frac{\overline{M}_{\rm P}}{\lambda}\left(\frac{\partial\lambda}{\partial\sigma}\right)\right], \qquad d_{\sigma} = m(f,\epsilon)\,\frac{\overline{M}_{\rm P}}{\lambda}\left(\frac{\partial\lambda}{\partial\sigma}\right),$$

$$m(f,\epsilon) = \frac{3(1-\epsilon)}{(1-2f)(4-2f-3\epsilon)}. \tag{5.23}$$

The function $m(f,\epsilon)$ depends on the slow roll parameter and on the growth rate in Hubble units, i.e. $f = \mathcal{F}/\mathcal{H}$. In the scale invariant case (i.e. $f=2$, see appendix B) $m(2,\epsilon) = 1/(3\,\epsilon)$. In the pure de Sitter case and for exactly scale invariant spectrum the integration over τ' would lead to logarithms of the conformal time coordinate which are absent in the quasi-de Sitter case. Equations (5.22) and (5.23) can be written in more explicit terms as:

$$c_{\varphi} = m(f,\epsilon)\left[\frac{\sqrt{2\epsilon}}{3} + \gamma_{\varphi}\right], \qquad d_{\varphi} = m(f,\epsilon)\,\gamma_{\varphi}, \tag{5.24}$$

$$c_{\sigma} = m(f,\epsilon)\left[\frac{2\epsilon}{3}\left(\frac{M}{\overline{M}_{\rm P}}\right) + \gamma_{\sigma}\left(\frac{\overline{M}_{\rm P}}{M}\right)\right], \qquad d_{\sigma} = m(f,\epsilon)\left(\frac{\overline{M}_{\rm P}}{M}\right)\gamma_{\sigma}. \tag{5.25}$$

The results of Eqs. (5.24) and (5.25) assume a simple parametrization of $\lambda(\varphi,\sigma)$, i.e.

$$\lambda(\varphi,\sigma) = \lambda_{*}\exp\left[\gamma_{\varphi}\frac{\varphi}{\overline{M}_{\rm P}} + \gamma_{\sigma}\frac{\sigma}{M}\right], \qquad \gamma_{\sigma} + \frac{\gamma_{\varphi}}{\sqrt{2\epsilon}} = \frac{(1-\epsilon)}{2\epsilon}\,f. \tag{5.26}$$

The second relation of Eq. (5.26) holds, strictly speaking, in the case of power-law inflation where $\eta = \epsilon$. It can be argued, however, that it remains valid in more general cases where $\epsilon \simeq \eta$.

5.3 Curvature perturbations

The curvature perturbations on comoving orthogonal hypersurfaces can be expressed in terms of Eqs. (5.19)–(5.20)

$$\mathcal{R}(\vec{k},\tau) \equiv -\frac{z_{\varphi}(\tau)\, q_{\varphi}(\vec{k},\tau) + z_{\sigma}(\tau)\, q_{\sigma}(\vec{k},\tau)}{z_{\varphi}^2(\tau) + z_{\sigma}^2(\tau)} \simeq -\frac{q_{\varphi}(\vec{k},\tau)}{z_{\varphi}(\tau)} - q_{\sigma}(\vec{k},\tau)\frac{z_{\sigma}(\tau)}{z_{\varphi}^2(\tau)}, \tag{5.27}$$

as it follows from Eqs. (4.5)–(4.7) by recalling that $z_{\varphi} = a\varphi'/\mathcal{H}$ and $z_{\sigma} = a\sigma'/\mathcal{H}$. The second equality of Eq. (5.27) follows in the limit $z_{\varphi}/z_{\sigma} = \varphi'/\sigma' \ll 1$ when σ is subdominant. The power spectrum of curvature perturbations is defined as

$$\langle \mathcal{R}(\vec{k},\tau)\, \mathcal{R}(\vec{p},\tau)\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}(k,\tau)\, \delta^{(3)}(\vec{k}+\vec{p}). \tag{5.28}$$

Bearing in mind Eqs. (3.26)–(3.28) and (5.17)–(5.18), Eqs. (5.19)–(5.20) can be inserted into Eq. (5.27) so that the explicit expression of $\mathcal{P}_{\mathcal{R}}(k,\tau)$ becomes

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k,\tau) &= \frac{k^3}{2\pi^2 z_{\varphi}^2}|F_{\varphi}(k,\tau)|^2 + \frac{k^3}{2\pi^2 z_{\sigma}^2}|F_{\sigma}(k,\tau)|^2\left(\frac{z_{\sigma}}{z_{\varphi}}\right)^4\\ &+ \mathcal{C}_{\varphi\sigma}^2(\tau)\,\frac{\mathcal{Q}_{\mathrm{B}}(k,\tau)}{H^4\,\overline{M}_{\mathrm{P}}^4} + \mathcal{D}_{\varphi\sigma}^2(\tau)\,\frac{\mathcal{Q}_{\mathrm{B}\Pi}(k,\tau)}{H^4\,\overline{M}_{\mathrm{P}}^4}.\end{aligned} \tag{5.29}$$

This expression shows that the power spectrum of curvature perturbations depends on the first-order correlations of the inflaton fluctuations as well as on the second-order correlations of the gauge fields whose related power spectra can be found in the appendix B.

The first two contributions to $\mathcal{P}_{\mathcal{R}}(k,\tau)$ appearing in Eq. (5.29) are the adiabatic contribution given by the inflaton and a generalized entropic contribution associated with the spectator field. Both $\mathcal{C}_{\varphi\sigma}(\tau)$ and $\mathcal{D}_{\varphi\sigma}(\tau)$ are slowly varying functions of τ (i.e. $\mathcal{C}_{\varphi\sigma}'(\tau) \simeq \mathcal{D}_{\varphi\sigma}(\tau) \simeq 0$) and are defined as

$$\mathcal{C}_{\varphi\sigma}^2(\tau) = \frac{\overline{M}_{\mathrm{P}}^2\, a^2(\tau)}{9\, z_{\varphi}^2(\tau)}\left[c_{\varphi} + \left(\frac{z_{\sigma}}{z_{\varphi}}\right)c_{\sigma}\right]^2, \qquad \mathcal{D}_{\varphi\sigma}^2(\tau) = \frac{\overline{M}_{\mathrm{P}}^2\, a^2(\tau)}{9\, z_{\varphi}^2(\tau)}\left[d_{\varphi} + \left(\frac{z_{\sigma}}{z_{\varphi}}\right)d_{\sigma}\right]^2. \tag{5.30}$$

What matters, for the present considerations, are those typical scales that had the longest time to grow and that left the Hubble radius at the onset of the inflationary phase even if, as we shall see, the beginning of inflation is essentially a free parameter related to the total number of inflationary efolds. The various contributions to the power spectrum must be compared the in the limit where the relevant scales are larger than the Hubble radius, i.e.

$$\mathcal{P}_{\mathrm{ad}}(k,\tau) = \frac{k^3}{2\pi^2 z_{\varphi}^2}|F_{\varphi}(k,\tau)|^2 = \frac{\mathcal{K}(\mu)}{8\pi^2\epsilon}\left(\frac{H}{\overline{M}_{\mathrm{P}}}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathrm{ad}}-1} \tag{5.31}$$

$$\mathcal{P}_{\mathrm{entr}}(k,\tau) = \frac{k^3}{2\pi^2 z_{\sigma}^2}|F_{\sigma}(k,\tau)|^2\left(\frac{z_{\sigma}}{z_{\varphi}}\right)^4 = \frac{\mathcal{K}(\tilde{\mu})}{4\pi^2\epsilon}\left(\frac{H}{\overline{M}_{\mathrm{P}}}\right)^2\left(\frac{M}{\overline{M}_{\mathrm{P}}}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathrm{entr}}-1}, \tag{5.32}$$

where, generically, $\mathcal{K}(z) = 2^{2z-1}\Gamma^2(z)/\pi$ so that $\mathcal{K}(3/2) = 1$ and[11]

$$n_{\rm ad} - 1 = 3 - 2\mu = -6\epsilon + 2\overline{\eta}, \qquad n_{\rm entr} - 1 = 3 - 2\tilde{\mu} = -2\epsilon. \tag{5.33}$$

Barring for the dependence of the spectral index on the slow roll corrections, we can clearly see that $\mathcal{P}_{\rm entr}(k,\tau) \sim \epsilon(M/\overline{M}_{\rm P})^2\,\mathcal{P}_{\rm ad}(k,\tau)$. Since $\epsilon < 1$ and $M \ll \overline{M}_{\rm P}$ the entropic contribution is strongly suppressed. If taken into account this component will lead to the kind of mixed initial conditions for CMB anisotropies often discussed in the literature [31, 32, 33, 34] also in the presence of large-scale magnetic fields [12].

In the simplest situation the total energy-momentum tensor of the system is conserved both before and after the transition between inflation and radiation [29, 30]. When the stress tensor undergoes a finite discontinuity on a space-like hypersurface the inhomogeneities are matched by requiring the continuity of the induced three metric and of the extrinsic curvature on that hypersurface. On uniform curvature hypersurfaces the continuity of the extrinsic curvature is guaranteed by the continuity of Δ_ϕ and Δ_β. This implies also the continuity of $\mathcal{R}$ as it can be explicitlly verified by solving the evolution equation of $\mathcal{R}$ (see Eq. (A.31)) valid in the postinflationary epoch. After the end of inflation the growth rate is zero and the evolution of curvature perturbations can be followed by means of a certain set of global variables. This discussion closely follows the considerations developed in Ref. [7].

In concluding this section it is appropriate to remark that Eqs. (4.8)–(4.9) and (5.9)–(5.10), even if deduced in a specific gauge, have a gauge-invariant meaning. The gauge-invariant generalization of the quasi-normal modes discussed in this section is given by

$$q_\varphi^{(\rm gi)} = a\chi_\varphi + z_\varphi\psi, \qquad q_\sigma^{(\rm gi)} = a\chi_\sigma + z_\sigma\psi. \tag{5.34}$$

Under the gauge transformation discussed prior to Eq. (4.2) χ_φ and χ_σ transform as $\chi_\varphi \to \overline{\chi}_\varphi - \varphi'\epsilon_0$ and $\chi_\sigma \to \overline{\chi}_\sigma - \sigma'\epsilon_0$. Thanks to Eq. (4.2) the quantities defined in Eq. (5.34) are left invariant.

The variables of Eq. (5.34) are the scalar field analog of the quantum excitations of an irrotational and relativistic fluid firstly discussed by Lukash [37] (see also [38, 39, 40]) right after one of the first formulations of inflationary dynamics [41]. The canonical normal mode identified in Ref. [37] is invariant under infinitesimal coordinate transformations as required in the context of the Bardeen formalism [42] (see also [38]). The subsequent analyses of Refs. [43] and [44] follow the same logic of [37] but in the case of scalar field matter; the normal modes of Refs. [37, 43, 44] coincide with the (rescaled) curvature perturbations on comoving orthogonal hypersurfaces [45, 46]. In the present case, as already pointed out, $q_\varphi^{(\rm gi)}$ is only a quasi-normal mode and becomes a truly normal mode only in the case when the spectator component vanishes.

[11] Recall Eqs. (5.14)–(5.15) and also the well known relations among the slow roll parameters, i.e. $\eta = \epsilon - \overline{\eta}$.

6 Growth rate of magnetic inhomogeneities

According to the requirements spelled out in [7], the predominance of the standard adiabatic mode over the magnetized contributions leads to a specific bound on the magnetic field intensity. This logic will now be applied to the curvature perturbations induced during the inflationary phase with the purpose of deriving accurate constraints on the growth rate of magnetized inhomogeneities in Hubble units.

6.1 Predominance of the adiabatic solution

Demanding that the adiabatic component is dominant against the entropic and the magnetic contributions, Eq. (5.29) implies

$$\mathcal{P}_{\mathrm{ad}}(k,\tau) > \mathcal{C}^{2}_{\varphi\sigma}(\tau)\,\frac{\mathcal{Q}_{\mathrm{B}}(k,\tau)}{H^{4}\,\overline{M}_{\mathrm{P}}^{4}} + \mathcal{D}^{2}_{\varphi\sigma}(\tau)\,\frac{\mathcal{Q}_{\mathrm{B}\Pi}(k,\tau)}{H^{4}\,\overline{M}_{\mathrm{P}}^{4}}, \tag{6.1}$$

having imposed $M \ll \overline{M}_{\mathrm{P}}$ and, consequently, $\mathcal{P}_{\mathrm{ad}}(k,\tau) \gg \mathcal{P}_{\mathrm{entr}}(k,\tau)$. The limit of Eq. (6.1) must be applied for the scales that experienced the largest amplification. For instance the galactic scale crossed the Hubble radius about 53 efolds prior to the end of inflation and had, therefore, less time to be amplified in comparison with the scales that left the horizon just at the beginning of inflation.

The bound of Eq. (6.1) is more constraining if imposed on the scales that crossed the Hubble radius just after the onset of inflation. Since the duration of inflation is unknown it is reasonable to take the total number of inflationary efolds N_{t} as a free parameter bounded, from below, by N_{max} denoting the maximal number of efolds that are accessible to our present observations (i.e. $N_{\mathrm{t}} \geq N_{\mathrm{max}}$). The value of N_{max} is derived by fitting the event horizon of the inflationary phase inside the present Hubble radius[12]

$$e^{N_{\mathrm{max}}} = (2\,\pi\,\epsilon\,\mathcal{A}_{\mathcal{R}}\;\Omega_{\mathrm{R}0})^{1/4}\,\left(\frac{M_{\mathrm{P}}}{H_{0}}\right)^{1/2}\left(\frac{H_{r}}{H}\right)^{\gamma-1/2}, \tag{6.2}$$

where the exponent γ controls the expansion rate during an intermediate phase ending at a putative scale H_r possibly much smaller than the Hubble rate during inflation denoted by H.

The parameters characterizing the dominant adiabatic component have been fixed to the values suggested by the best fit to the WMAP9 data [47] analyzed in terms of the vanilla ΛCDM model; this corresponds, in particular, to $n_{\mathrm{ad}} = 0.972$ and $\mathcal{A}_{\mathcal{R}} = (2.41 \pm 0.10) \times 10^{-9}$. Different data sets, like for instance the WMAP7 data [48, 49] would imply $n_{\mathrm{ad}} = 0.963$ and $\mathcal{A}_{\mathcal{R}} = (2.43 \pm 0.11) \times 10^{-9}$. These differences are immaterial for the present considerations.

For consistency with big-bang nucleosynthesis H_r, in Eq. (6.2) can be, at most, $10^{-44} M_{\mathrm{P}}$ corresponding to a reheating scale occurring just prior to the formation of the light nuclei. If

[12]For numerical estimates we recall that $h_0^2 \Omega_{\mathrm{R}0} = 4.15 \times 10^{-5}$; the present value of the Hubble rate $H_0 = 100\, h_0\, \mathrm{Mpc}^{-1}\, \mathrm{km/sec}$ in Planck units is $H_0 = 1.22 \times 10^{-6}\, (h_0/0.7)\, M_{\mathrm{P}}$.

$\gamma - 1/2 > 0$ (as it happens if $\gamma = 2/3$ when the postinflationary background is dominated by dust) $N_{\max}$ diminishes in comparison with the case when $H = H_r$. Conversely if $\gamma - 1/2 < 0$ (as it happens in $\gamma = 1/3$ when the postinflationary background is dominated by stiff sources) $N_{\max}$ increases. If $H_r = H$ (or if $\gamma = 1/2$) there is a sudden transition between the inflationary and the postinflationary regimes and, in this case, we have approximately $N_{\max} \simeq 64 + 0.25 \ln \epsilon$.

In the case case of a standard postinflationary history $N_{\max}$ coincides with the number of efolds necessary to address the conventional drawbacks of the hot big bang model [50, 51]. Whenever $N_{\rm t} > N_{\max}$ the redshifted value of the inflationary event horizon exceeds the present value of the Hubble radius. If $N_{\rm t} = N_{\max}$ the scales which were still larger than the

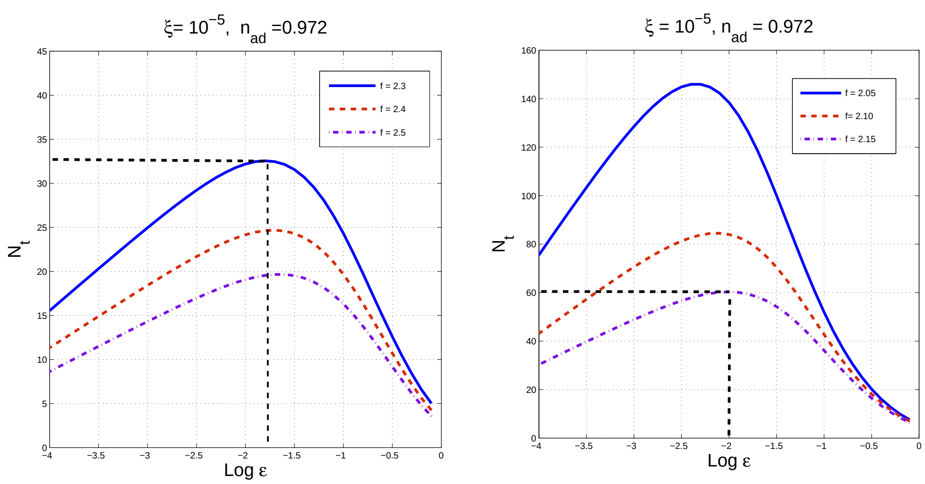


Figure 1: The bound on the growth rate is illustrated by plotting the total number of efolds as a function of the slow roll parameter. The allowed region in the parameter space is below each of the various curves corresponding to different values of f. In these plots as well as in Figs. 2 we have set $M = 10^{-4}\overline{M}_{\rm P}$.

Hubble radius around matter-radiation equality left the inflationary Hubble radius about $N_{\max}$ efolds prior to the end of inflation at least for a standard postinflationary history. Consequently the most constraining bound derivable from Eq. (6.1) is achieved by demanding a typical number of efolds close to $N_{\max}$ for comoving scale of the order of $q_{\rm p}$ denoting the pivot wavenumber at which the amplitude of the curvature power spectrum is commonly assigned when analyzing the temperature and polarization anisotropies [47, 48, 49].

Using the results of appendix B and of Sec. 5, Eq. (6.1) can be phrased as in terms of f (i.e. the growth rate of the magnetized inhomogeneities expressed in Hubble units). Besides the growth rate in Hubble units and the total number of efolds, the parameter ξ measures the Hubble rate in Planck units [13] and is given by $\xi = H/\overline{M}_{\rm P}$. If the adiabatic mode is the

[13]Some authors denote with ξ a further slow roll parameter containing four derivatives of the inflaton potential. The notations used here are different and, with this remark, no confusion is possible.

only source of inhomogeneity then there is a specific relation between ξ, ϵ and the amplitude of curvature perturbations at the pivot scale q_{p}. In the latter case and using the WMAP9 [47] data we have[14]

$$\xi = \frac{H}{\overline{M}_{\mathrm{P}}} = \pi\,\sqrt{8\,\epsilon\,\mathcal{A}_{\mathcal{R}}}, \quad \mathcal{A}_{\mathcal{R}} = (2.41 \pm 0.10) \times 10^{-9}. \tag{6.3}$$

In what follows ξ will be taken as a free parameter. The values of ξ are assigned independently of the values of ϵ in all the figures of this section except for Fig. 3 holding in the case of Eq. (6.3) where $\xi \propto \sqrt{\epsilon}$. The inequality (6.1) cannot be simply inverted in terms of f or in terms of the slow roll parameters. Equation (6.1) can instead be written as

$$\frac{\mathcal{K}(\mu)}{8\pi^2\epsilon}\xi^2\left(\frac{q_{\mathrm{p}}}{aH}\right)^{n_{\mathrm{ad}}-1} > \mathcal{M}(f,\epsilon,\eta,\gamma_{\varphi},\gamma_{\sigma},M)\,\xi^4\,e^{N_{\mathrm{t}}g_B(f,\epsilon)}, \tag{6.4}$$

where the function $\mathcal{M}(f,\epsilon,\eta,\gamma_{\varphi},\gamma_{\sigma},M)$ is

$$\mathcal{C}^2_{\varphi\sigma}(f,\epsilon,\eta,\gamma_{\varphi},\gamma_{\sigma},M)\mathcal{C}_{\mathrm{B}}(f,\epsilon)\mathcal{L}_{\mathrm{B}}(f,\epsilon,q_{\mathrm{p}})+\mathcal{D}^2_{\varphi\sigma}(f,\epsilon,\eta,\gamma_{\varphi},\gamma_{\sigma},M)\mathcal{C}_{\mathrm{B}\Pi}(f,\epsilon)\mathcal{L}_{\mathrm{B}\Pi}(f,\epsilon,q_{\mathrm{p}}). \tag{6.5}$$

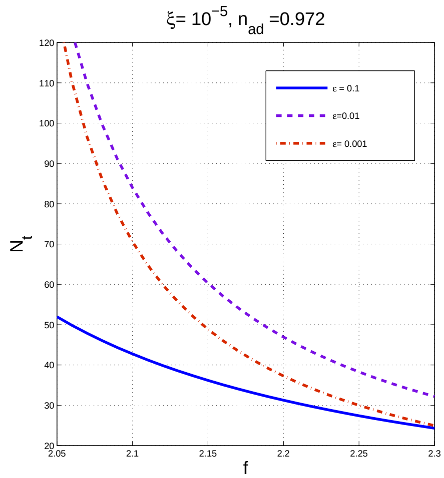
As already mentioned the various contributions to Eq. (6.5) can be found in Eqs. (5.30), (B.16) and (B.18)–(B.19).

6.2 Illustration of the constraints

Whenever the growth rate exceeds a critical value (for a given duration of the inflationary N_{t} and for a fixed value of the other parameters), the inequalities of Eqs. (6.1)–(6.4) are first saturated and then violated. With the aim of an accurate determination of the critical rate, the attention shall be first focussed on the case $\gamma_{\varphi} = 0$; in this case, as discussed in Sec. 5, the growth of the magnetic inhomogeneities is only due to the spectator field. In Fig. 1 the values of N_{t} are illustrated for different rates f as a function of ϵ. The allowed region in the parameter space is below the various curves of the two plots. The vertical and horizontal dashed lines in the left plot of Fig. 1 correspond to a value $f = 2.3$ (for $\epsilon \simeq 0.01$) forbidding any reasonable duration of the inflationary phase since N_{t} must be smaller than about 35. Larger values of f would be even more constraining for N_{t}; we conclude that the range of physical values is $2 \leq f < 2.3$. For $f < 2$ the growth rate is not constrained by the predominance of the adiabatic mode since the magnetic energy density decreases (rather than increasing) for typical wavelengths larger than the Hubble radius.

In Fig. 1 (plot at the right) the region of parameter space $2.05 \leq f \leq 2.15$ is more accurately scrutinized. As the vertical and horizontal dashed lines indicate, for $f \simeq 2.15$ and $\epsilon \simeq 0.01$ we are really on the borderline of the allowed region: as soon as $f > 2.15$, the

[14]To avoid confusions we remind that we used throughout $\overline{M}_{\mathrm{P}} = M_{\mathrm{P}}/\sqrt{8\pi}$. Some authors prefer to use M_{P} instead of $\overline{M}_{\mathrm{P}}$ and, in this case, the analog of Eq. (6.3) reads $(H/M_{\mathrm{P}}) = \sqrt{\pi\,\epsilon\mathcal{A}_{\mathcal{R}}}$.



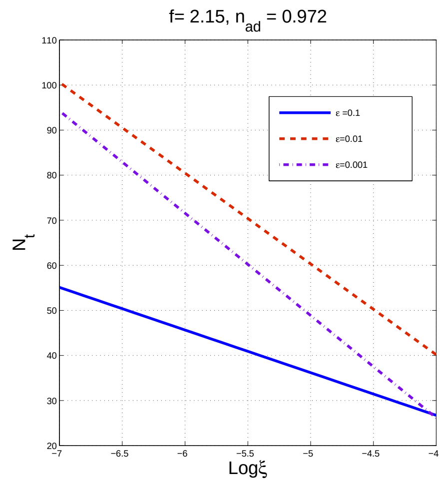


Figure 2: The bound on the growth rate is illustrated in the planes $(N_{\rm t}, f)$ and $(N_{\rm t}, \xi)$. As in Fig. 1, the allowed region in the parameter space lies below the corresponding curve.

total number of allowed efolds drops below 60 that is insufficient to address and solve, for instance, the horizon problem of the conventional hot big bang model [50, 51]

The same point is analyzed, within a complementary perspective in Fig. 2. A value $f = 2.2$ in the left plot of Fig. 2 corresponds roughly to $N_{\rm t} \simeq 30$ (for $\epsilon = 0.1$). This means that the achievable number of efolds cannot be as large as $N_{\rm max}$: even for smaller values of ϵ it turns out that $N_{\rm t} \leq 50$.

In the right plot of Fig. 2 the bounds on the growth rate are illustrated for $f = 2.15$ but in the plane $(\xi,\, N_{\rm t})$. The line $N_{\rm t} = 60$ crosses the dashed line (corresponding to $\epsilon = 0.01$) for $\xi \simeq 10^{-5}\,\overline{M}_{\rm P}$. Smaller values of ξ would correspond to inflationary phases occurring at low curvature. In this case $N_{\rm t}$ can be larger (for the same range of growth rates) but the adiabatic mode will not be able to account for the observed temperature and polarization anisotropies probed by direct CMB observations. The constraints set by the growth of the inhomogeneities are stronger than the ones simply implied by the backreaction. Given a growth rate f of the magnetic field we have to demand $(\overline{\rho}_{\rm E} + \overline{\rho}_{\rm B}) < 3H^2\,\overline{M}_{\rm P}^2$; the latter condition, already discussed in Sec. 2 can be written explicitly in the case of a monotonic growth rate:

$$\frac{H^4}{32\pi^2}\int_{x_{\rm min}}^{x_{\rm max}} dx\, x^4 \biggl[|H_{\nu}^{(1)}(x)|^4 + |H_{\nu-1}^{(1)}(x)|^4\biggr] < 3H^2\,\overline{M}_{\rm P}^2, \tag{6.6}$$

where $\nu = 1/2 + f(1+\epsilon)$ for $\epsilon < 1$ and where $x_{\rm min} = k_{\rm min}\tau$ while $x_{\rm max} = k_{\rm max}\tau$. The integration can be separated in the region $x > 1$ (where the energy density decreases as a^{-4}) and the region $x < 1$ where the magnetic energy density increases while the electric energy density still decreases. By demanding that $k_{\rm min} = 1/\tau_{\rm min}$ is the first scale leaving the Hubble radius at the onset of inflation and that $k_{\rm max} = 1/\tau_{\rm max}$ is the last scale leaving the Hubble radius at the end of inflation, the net growth of the energy density can be constrained. The

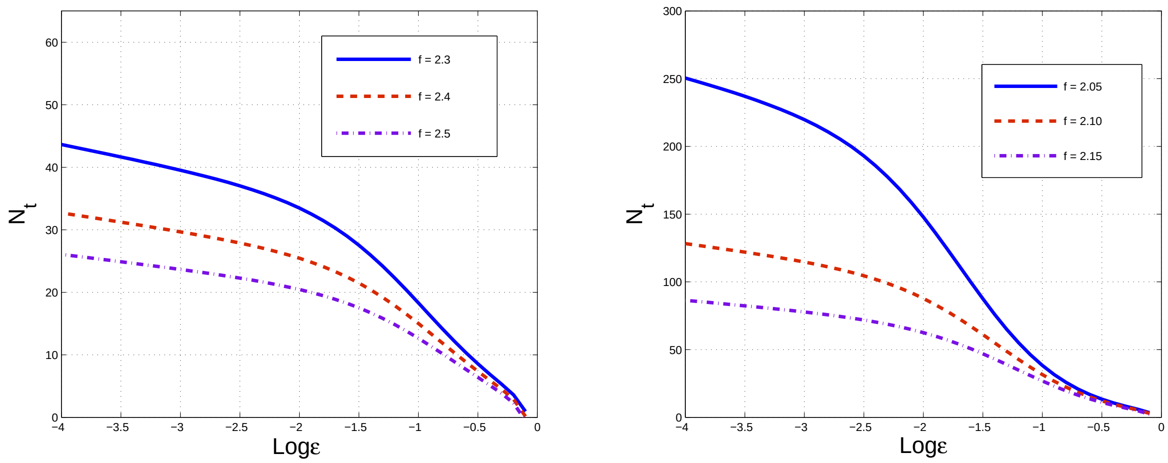


Figure 3: The constraints stemming from the requirement that the energy density of the amplified magnetic field is subdominant in comparison with the energy density of the background geometry. The values of ξ have been fixed as in Eq. (6.3).

results are illustrated in Fig. 3 in the usual plane $(N_{\rm t}, \epsilon)$.

The results of Fig. 3 are less restrictive than the requirements obtained from the growth of the inhomogeneities. For instance, the full line of the right plot of Fig. 1 has a maximum for $N_{\rm t} \simeq 140$ while the full line of the right plot of Fig. 3 seems to allow for more than 200 efolds. Backreaction effects can very well be under control but the inhomogeneities may grow larger than the adiabatic contribution. If we commit ourselves to a specific scenario, the magnetic energy density must not affect the equation of the spectator field. Adopting the parametrization previously discussed. This condition would demand $\gamma_{\sigma}/M(\overline{\rho}_{\rm B} - \overline{\rho}_{\rm E}) < 3H\dot{\sigma}$ implying

$$\frac{\gamma_{\sigma}\,(\overline{\rho}_{\rm B} - \overline{\rho}_{\rm E})}{3H^2\overline{M}_{\rm P}^2} < \frac{M}{\overline{M}_{\rm P}} < 1. \tag{6.7}$$

Let us consider, as an example, the case $\gamma_{\varphi} = 0$. In this case γ_{σ} is directly expressible in terms of f and ϵ according to Eq. (5.26). The condition (6.7) can then be plotted for different values of $N_{\rm t}$, ϵ and f with the same logic leading to Figs. 1 and 2. As it can be explicitly seen the inequalities (6.7) are satisfied with $M = 1.5 \times 10^{-2}\overline{M}_{\rm P}$ for $N_{\rm t} \simeq 65$ and $10^{-4} < \epsilon < 10^{-2}$. We therefore conclude that the predominance of the adiabatic mode represents a more constraining criterion than the simple backreaction requirements.

Let us now recall that the case $\gamma_{\sigma} = 0$, according to some considerations, would be less plausible since, in this case, the inflaton would be directly coupled to the gauge fields and the flatness of the potential might be in danger. In spite of these caveats, in Fig. 4 the case $\gamma_{\sigma} = 0$ is illustrated and should be compared with Fig. 1 (obtained in the case $\gamma_{\varphi} = 0$). Provided $f \leq 2.15$ and $0.001 \leq \epsilon \leq 0.1$ the total number of efolds is larger than 65. Conversely, larger values of the growth rate (i.e. $f > 2.2$) constrain the number of efolds to

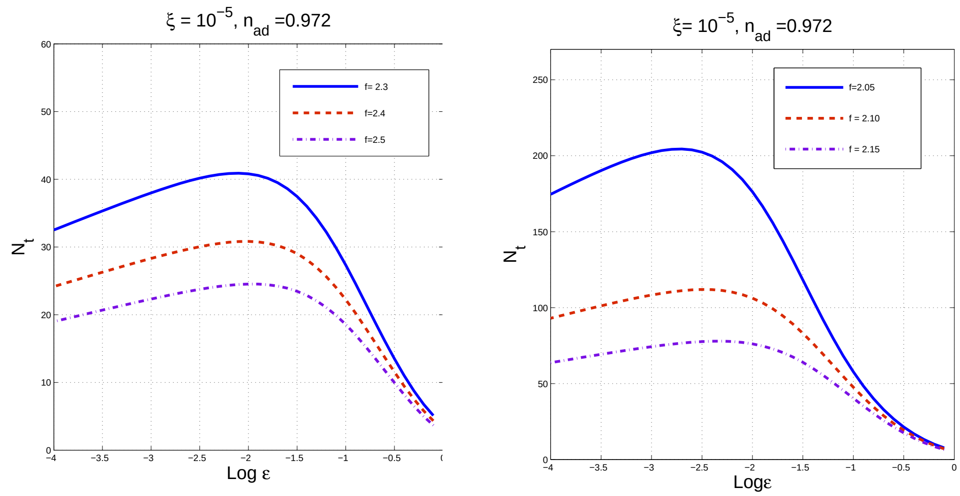


Figure 4: The bound on the growth rate is illustrated in the plane ($N_{\rm t}$,ϵ). This figure is the analog of Fig. 1 but in the case $\gamma_{\sigma} = 0$.

be smaller than 65 and are therefore not acceptable.

6.3 Concluding remarks

The bounds on the growth rate can be also translated into constraints on the magnetic spectral index entering the phenomenological discussion of the effects of predecoupling magnetic fields on the temperature and polarization anisotropies. The magnetic spectral index defined in [52] gives the slope of the magnetic field spectrum, i.e. $P_{\rm B} \propto k^{n_{\rm B}-1}$. The relation between f and $n_{\rm B}$ is $n_{\rm B} = 5 - 2f(1+\epsilon)$. The bounds on f derived in this section then imply a lower bound on $n_{\rm B} > 0.6 - 4.4\epsilon$. It is interesting to notice that the results of [52] are compatible with this limit: in a frequentistic perspective the analysis of the temperature and polarization correlations in the magnetized ΛCDM scenario implies that values $n_{\rm B} < 0.9$ are excluded to 95% confidence level. The present findings suggest that a nearly scale-invariant magnetic field spectrum induced by inflationary magnetogenesis is compatible with the range $2 \leq f < 2.2$ pinned down by requiring the predominance of the adiabatic mode during conventional inflation as argued some time ago in different contexts (see, for instance, [21, 27]). In the nearly scale-invariant case the amplitude of the physical magnetic power spectrum is of the order of 1.44×10^{-11} G (for $\epsilon = 0.01$ and $\mathcal{A}_{\mathcal{R}} = 2.41 \times 10^{-9}$) at the epoch of the gravitational collapse of the protogalaxy.

Acknowledgments

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A Evolution equations in uniform curvature gauge

The main set of governing equations used to derive various results discussed in the bulk of the paper will be illustrated in detail. With the gauge choice (4.4) the coordinate system is totally fixed without the need of further conditions: because of this property the functions $\phi(\vec{x},\tau)$ and $\beta(\vec{x},\tau)$ bear an extremely simple relation to one of the conventional sets of gauge-invariant variables. In the gauge (4.4) the inhomogeneities of the energy-momentum tensors $T_{\alpha}^{\beta}(\varphi)$, $T_{\alpha}^{\beta}(\sigma)$ and $\mathcal{T}_{\alpha}^{\beta}(\rho,\, p)$ are, respectively,

$$\delta_{\rm s}\, T_{0}^{0}(\varphi) = \delta\rho_{\varphi}, \qquad \delta_{\rm s}\, T_{0}^{i}(\varphi) = -\frac{\varphi'^{2}}{a^2}\biggl(\frac{\partial^{i}\chi_{\varphi}}{\varphi'} + \partial^{i}\beta\biggr), \quad \delta_{\rm s}\, T_{i}^{j}(\varphi) = -\delta p_{\varphi}\delta_{i}^{j}, \tag{A.1}$$

$$\delta_{\rm s}\, T_{0}^{0}(\sigma) = \delta\rho_{\sigma}, \qquad \delta_{\rm s}\, T_{0}^{i}(\sigma) = -\frac{\sigma'^{2}}{a^2}\biggl(\frac{\partial^{i}\chi_{\sigma}}{\sigma'} + \partial^{i}\beta\biggr), \quad \delta_{\rm s}\, T_{i}^{j}(\sigma) = -\delta p_{\sigma}\,\delta_{i}^{j}, \tag{A.2}$$

$$\delta_{\rm s}\mathcal{T}_{0}^{0} = \delta\rho, \quad \delta_{\rm s}\mathcal{T}_{i}^{j} = -\delta p\,\delta_{i}^{j}, \quad \delta_{\rm s}\mathcal{T}_{0}^{i} = (p+\rho)v^{i}, \tag{A.3}$$

where χ_{φ} and χ_{σ} denote, respectively, the fluctuations of the inflaton φ and of the spectator field σ; v^{i} denotes the three-velocity of the fluid in the gauge (4.4). The explicit expressions of $(\delta\rho_{\varphi},\, \delta p_{\varphi})$ and of $(\delta\rho_{\sigma},\, \delta p_{\sigma})$ are[15], respectively:

$$\delta\rho_{\varphi} = \frac{1}{a^2}\biggl(-\phi\varphi'^{2} + \chi_{\varphi}'\varphi' + a^2\frac{\partial V}{\partial\varphi}\chi_{\varphi}\biggr), \quad \delta p_{\varphi} = \frac{1}{a^2}\biggl(-\phi\varphi'^{2} + \chi_{\varphi}'\varphi' - a^2\frac{\partial V}{\partial\varphi}\chi_{\varphi}\biggr), \tag{A.4}$$

$$\delta\rho_{\sigma} = \frac{1}{a^2}\biggl(-\phi\sigma'^{2} + \chi_{\sigma}'\sigma' + a^2\frac{\partial W}{\partial\sigma}\chi_{\sigma}\biggr), \quad \delta p_{\sigma} = \frac{1}{a^2}\biggl(-\phi\sigma'^{2} + \chi_{\sigma}'\sigma' - a^2\frac{\partial W}{\partial\sigma}\chi_{\sigma}\biggr). \tag{A.5}$$

The fluctuations of the Einstein tensor $\mathcal{G}_{\mu}^{\nu} = R_{\mu}^{\nu} - R\,\delta_{\mu}^{\nu}/2$, always in the gauge (4.4), are instead:

$$\delta_{\rm s}\mathcal{G}_{0}^{0} = \frac{2}{a^2}\biggl[-\mathcal{H}\nabla^2\beta - 3\mathcal{H}^2\,\phi\biggr], \qquad \delta_{\rm s}\mathcal{G}_{0}^{i} = \frac{2}{a^2}\partial^{i}\biggl[-\mathcal{H}\phi + (\mathcal{H}' - \mathcal{H}^2)\beta\biggr], \tag{A.6}$$

$$\begin{aligned} \delta_{\rm s}\mathcal{G}_{i}^{j} &= \frac{1}{a^2}\biggl\{\biggl[-2(\mathcal{H}^2 + 2\mathcal{H}')\phi - 2\mathcal{H}\phi'\biggr] - \nabla^2\biggl(\phi + \beta' + 2\mathcal{H}\beta\biggr)\biggr\}\delta_{i}^{j} \\ &+ \frac{1}{a^2}\partial_{i}\partial^{j}\biggl(\beta' + 2\mathcal{H}\beta + \phi\biggr), \end{aligned} \tag{A.7}$$

The combination of Eqs. (A.1)–(A.2) and (A.3) with Eqs. (A.6)–(A.7) implies that the (00) and $(0i)$ components of the perturbed Einstein equations with mixed indices become[16]:

$$\mathcal{H}\nabla^2\beta + 3\mathcal{H}^2\phi = -4\pi G a^2\biggl[\delta\rho_{\rm t} + \delta\rho_{\rm B} + \delta\rho_{\rm E}\biggr], \tag{A.8}$$

$$(\mathcal{H}' - \mathcal{H}^2)\nabla^2\beta - \mathcal{H}\nabla^2\phi = 4\pi G a^2\biggl[(p+\rho)\theta + P + (p_{\varphi} + \rho_{\varphi})\theta_{\varphi} + (p_{\sigma} + \rho_{\sigma})\theta_{\sigma}\biggr], \tag{A.9}$$

[15]To avoid lengthy notations we wrote $\delta\rho_{\varphi}$ and $\delta\rho_{\sigma}$ (instead of $\delta_{\rm s}\rho_{\varphi}$ and $\delta_{\rm s}\rho_{\sigma}$), $\delta\rho$ (instead of $\delta_{\rm s}\rho$) and similarly for the corresponding pressures; this notation is fully justified and unambiguous once the scalar nature of the fluctuations has been established, as specified by the general formulae written above.

[16]Equations (A.8) and (A.9) are commonly referred to as, respectively, the Hamiltonian and the momentum constraints.

where $\delta\rho_{\rm t} = \delta\rho + \delta\rho_{\varphi} + \delta\rho_{\sigma}$ and

$$\theta_{\varphi} = -\frac{\Delta_{\varphi}}{\varphi'} - \Delta_{\beta}, \quad \theta_{\sigma} = -\frac{\Delta_{\sigma}}{\sigma'} - \Delta_{\beta}, \quad \theta(\vec{x},\tau) = \partial_{i}\, v^{i}. \tag{A.10}$$

In Eq. (A.10) the following practical notations

$$\Delta_{\varphi} = \nabla^2 \chi_{\varphi}, \qquad \Delta_{\sigma} = \nabla^2 \chi_{\sigma}, \qquad \Delta_{\beta} = \nabla^2 \beta, \tag{A.11}$$

have been introduced. The (ij) component of the perturbed Einstein equations reads:

$$\begin{aligned}&\biggl[-(\mathcal{H}^2 + 2\mathcal{H}')\phi - \mathcal{H}\phi' - \frac{1}{2}\nabla^2(\phi + \beta' + 2\mathcal{H}\beta)\biggr]\delta_{i}^{j} + \frac{1}{2}\partial_{i}\partial^{j}\biggl[\phi + \beta' + 2\mathcal{H}\beta\biggr]\\ &= 4\pi G a^2\biggl\{-\biggl[\delta p_{\rm t} + \delta p_{\rm B} + \delta p_{\rm E}\biggr]\delta_{i}^{j} + \Pi_{i}^{j}(E) + \Pi_{i}^{j}(B)\biggr\},\end{aligned} \tag{A.12}$$

where, in full analogy with Eq. (A.10), the total pressure fluctuation $\delta p_{\rm t}$ has been defined:

$$\delta p_{\rm t} = \delta p + \delta p_{\varphi} + \delta p_{\sigma}. \tag{A.13}$$

The separation of the traceless part from the trace in Eq. (A.12) implies the following pair of relations:

$$(\mathcal{H}^2 + 2\mathcal{H}')\phi + \mathcal{H}\phi' + \frac{1}{3}\nabla^2(\phi + \beta' + 2\mathcal{H}\beta) = 4\pi G a^2(\delta p_{\rm t} + \delta p_{\rm B} + \delta p_{\rm E}), \tag{A.14}$$

$$\partial_{i}\partial^{j}[\phi + \beta' + 2\mathcal{H}\beta] - \frac{1}{3}\nabla^2[\phi + \beta' + 2\mathcal{H}\beta]\delta_{i}^{j} = 8\pi G a^2\biggl[\Pi_{i}^{j}(E) + \Pi_{i}^{j}(B)\biggr], \tag{A.15}$$

that can be further simplified by recalling Eq. (3.25):

$$(\mathcal{H}^2 + 2\mathcal{H}')\Delta_{\phi} + \mathcal{H}\Delta_{\phi}' = 4\pi G a^2\biggl[\nabla^2(\delta p + \delta p_{\varphi} + \delta p_{\sigma}) - \nabla^2\Bigl(\Pi_{\rm E} + \Pi_{\rm B}\Bigr)\biggr], \tag{A.16}$$

$$\Delta_{\beta}' + 2\mathcal{H}\Delta_{\beta} + \Delta_{\phi} = 12\pi G a^2\Bigl(\Pi_{\rm E} + \Pi_{\rm B}\Bigr). \tag{A.17}$$

In Eq. (A.17) the same notations established in Eq. (A.11) have been employed. During inflation the perturbative variables necessary to describe the evolution of the whole system are then given by Δ_{ϕ}, Δ_{β}, Δ_{φ} and Δ_{σ}. Neglecting the fluid sources, the Hamiltonian and momentum constraints of Eqs. (A.8) and (A.9) can then be written as

$$\mathcal{H}\nabla^2\Delta_{\beta} + 3\mathcal{H}^2\Delta_{\phi} = -4\pi G a^2\biggl[\nabla^2(\delta\rho_{\varphi} + \delta\rho_{\sigma}) + \nabla^2(\delta\rho_{\rm B} + \delta\rho_{\rm E})\biggr], \tag{A.18}$$

$$(\mathcal{H}' - \mathcal{H}^2)\Delta_{\beta} - \mathcal{H}\Delta_{\phi} = 4\pi G a^2\biggl\{P - \frac{1}{a^2}\biggl[\varphi'\Delta_{\varphi} + \sigma'\Delta_{\sigma} + ({\varphi'}^2 + {\sigma'}^2)\Delta_{\beta}\biggr]\biggr\}. \tag{A.19}$$

The evolution equations of Δ_{φ} and of Δ_{σ} are derived from the perturbed version of Eqs. (2.6) and (2.7)

$$\Delta_{\varphi}'' + 2\mathcal{H}\Delta_{\varphi}' - \nabla^2\Delta_{\varphi} + \frac{\partial^2 V}{\partial\varphi^2}a^2\Delta_{\varphi} + 2\frac{\partial V}{\partial\varphi}a^2\,\Delta_{\phi}$$

$$-\varphi'(\Delta'_{\phi}+\nabla^2\Delta_{\beta}) = \frac{a^2}{\lambda}\frac{\partial\lambda}{\partial\varphi}\nabla^2(\delta\rho_{\rm E}-\delta\rho_{\rm B}), \qquad (A.20)$$

$$\Delta''_{\sigma}+2\mathcal{H}\Delta'_{\sigma}-\nabla^2\Delta_{\sigma}+\frac{\partial^2 W}{\partial\sigma^2}a^2\Delta_{\sigma}+2\frac{\partial W}{\partial\sigma}a^2\,\Delta_{\phi}$$
$$-\sigma'(\Delta'_{\phi}+\nabla^2\Delta_{\beta}) = \frac{a^2}{\lambda}\frac{\partial\lambda}{\partial\sigma}\nabla^2(\delta\rho_{\rm E}-\delta\rho_{\rm B}). \qquad (A.21)$$

The system of Eqs. (A.20) and (A.21) can be reduced to a set of quasi-normal modes whose evolution equations are mutually coupled but decoupled from all other perturbation variables. The sum of these quasi-normal modes, weighted by coefficients that depend on the geometry, gives the curvature perturbations as explained in Eqs. (4.6)–(5.27).

The evolution equations for Δ_{φ} and Δ_{σ} can be decoupled from the remaining perturbation variables as follows. From Eq. (A.9), neglecting the fluid component, we obtain an expression for Δ_{ϕ}; a derivation with respect to the conformal time coordinate will give Δ'_{ϕ}. The final result of this step is

$$\Delta_{\phi} = 4\pi G\Big[\Big(\frac{\varphi'}{\mathcal{H}}\Big)\Delta_{\varphi}+\Big(\frac{\sigma'}{\mathcal{H}}\Big)\Delta_{\sigma}\Big]-\frac{4\pi G a^2}{\mathcal{H}}P, \qquad (A.22)$$

$$\Delta'_{\phi} = 4\pi G\Big[\Big(\frac{\varphi'}{\mathcal{H}}\Big)\Delta'_{\varphi}+\Big(\frac{\varphi'}{\mathcal{H}}\Big)'\Delta_{\varphi}+\Big(\frac{\sigma'}{\mathcal{H}}\Big)\Delta'_{\sigma}+\Big(\frac{\sigma'}{\mathcal{H}}\Big)'\Delta_{\sigma}\Big]$$
$$-\frac{4\pi G a^2}{\mathcal{H}}\Big[P'+\Big(2\mathcal{H}-\frac{\mathcal{H}'}{\mathcal{H}}\Big)P\Big]. \qquad (A.23)$$

From the Hamiltonian constraint of Eq. (A.8), always during the inflationary phase, we can obtain $\nabla^4\beta=\nabla^2\Delta_{\beta}$ and eliminate, in the derived expression, Δ_{ϕ} through Eq. (A.22). This algebraic step leads to three typical terms: the first one contains the dependence on Δ_{φ} and Δ_{σ}; the second term contains Δ'_{φ} and Δ'_{σ}; the third term depends on P, P' and $(\delta\rho_{\rm B}+\delta\rho_{\rm E})$. The full expression of $\nabla^2\Delta_{\beta}$ is:

$$\nabla^2\Delta_{\beta} = -4\pi G\Big\{\Big[\Big(2\mathcal{H}+\frac{\mathcal{H}'}{\mathcal{H}}\Big)\Big(\frac{\varphi'}{\mathcal{H}}\Big)+\frac{a^2}{\mathcal{H}}\frac{\partial V}{\partial\varphi}\Big]\Delta_{\varphi}+\Big[\Big(2\mathcal{H}+\frac{\mathcal{H}'}{\mathcal{H}}\Big)\Big(\frac{\sigma'}{\mathcal{H}}\Big)+\frac{a^2}{\mathcal{H}}\frac{\partial W}{\partial\sigma}\Big]\Delta_{\sigma}\Big\}$$
$$-4\pi G\Big[\Big(\frac{\varphi'}{\mathcal{H}}\Big)\Delta'_{\varphi}+\Big(\frac{\sigma'}{\mathcal{H}}\Big)\Delta'_{\sigma}\Big]+\frac{4\pi G a^2}{\mathcal{H}}\Big[\Big(2\mathcal{H}+\frac{\mathcal{H}'}{\mathcal{H}}\Big)P-\nabla^2(\delta\rho_{\rm B}+\delta\rho_{\rm E})\Big]. \qquad (A.24)$$

By then summing up term by term Eqs. (A.23) and (A.24) the term $(\Delta'_{\phi}+\nabla^4\beta)$ that appears in Eqs. (A.20) and (A.21) can be explicitly obtained:

$$\nabla^2\Delta_{\beta}+\Delta'_{\phi} = -4\pi G\Big\{\Big[2\Big(2\mathcal{H}+\frac{\mathcal{H}'}{\mathcal{H}}\Big)\Big(\frac{\varphi'}{\mathcal{H}}\Big)+2\frac{a^2}{\mathcal{H}}\frac{\partial V}{\partial\varphi}\Big]\Delta_{\varphi}$$
$$+\Big[2\Big(2\mathcal{H}+\frac{\mathcal{H}'}{\mathcal{H}}\Big)\Big(\frac{\sigma'}{\mathcal{H}}\Big)+2\frac{a^2}{\mathcal{H}}\frac{\partial W}{\partial\sigma}\Big]\Delta_{\sigma}+\frac{a^2}{\mathcal{H}}\Big[P'-2\frac{\mathcal{H}'}{\mathcal{H}}P+\nabla^2(\delta\rho_{\rm B}+\delta\rho_{\rm E})\Big]\Big\}. \qquad (A.25)$$

With the aid of Eq. (A.25) and of the other equations derived in this appendix, Eqs. (A.20) and (A.21) reduce to Eqs. (4.8) and (4.9). As anticipated these equations are mutually coupled but decoupled from all other perturbations variables.

After a transition regime, in the postinflationary phase, the coupling of the sources to the growth rate of the magnetic and electric fields disappears. The covariant conservation equation of energy-momentum tensor reduces to

$$\delta\rho_{\mathrm{t}}' + (p_{\mathrm{t}} + \rho_{\mathrm{t}})\theta_{\mathrm{t}} + 3\mathcal{H}(\delta p_{\mathrm{t}} + \delta\rho_{\mathrm{t}}) = 0, \tag{A.26}$$

while the equation for the three-divergence of the total fluid velocity becomes:

$$(\theta_{\mathrm{t}} + \Delta_\beta)' + \frac{[p_{\mathrm{t}}' + \mathcal{H}(p_{\mathrm{t}} + \rho_{\mathrm{t}})]}{(p_{\mathrm{t}} + \rho_{\mathrm{t}})}(\theta_{\mathrm{t}} + \Delta_\beta) + \frac{\nabla^2 \delta p_{\mathrm{t}}}{(p_{\mathrm{t}} + \rho_{\mathrm{t}})} + \Delta_\phi = 0. \tag{A.27}$$

Using Eq. (A.26) and recalling that $\zeta = (\delta\rho_{\mathrm{t}} + \delta\rho_{\mathrm{B}} + \delta\rho_{\mathrm{E}})/[3(\rho_{\mathrm{t}} + p_{\mathrm{t}})]$ the evolution of ζ becomes

$$\begin{aligned} \zeta' &= -\frac{\mathcal{H}}{(p_{\mathrm{t}} + \rho_{\mathrm{t}})}\delta p_{\mathrm{nad}} + \frac{\mathcal{H}}{p_{\mathrm{t}} + \rho_{\mathrm{t}}}\left(c_{\mathrm{st}}^2 - \frac{1}{3}\right)(\delta\rho_{\mathrm{B}} + \delta\rho_{\mathrm{E}}) \\ &- \frac{P}{3(p_{\mathrm{t}} + \rho_{\mathrm{t}})} - \frac{\vec{J}\cdot\vec{E}}{3(p_{\mathrm{t}} + \rho_{\mathrm{t}})a^4} - \frac{\theta_{\mathrm{t}}}{3}, \end{aligned} \tag{A.28}$$

where $\delta p_{\mathrm{nad}} = \delta p_{\mathrm{t}} - c_{\mathrm{st}}^2 \delta\rho_{\mathrm{t}}$. Similarly, the evolution equation for $\mathcal{R}$ can be almost immediately obtained by subtracting Eq. (A.8) (mutiplied by c_{st}^2) from Eq. (A.14) and by recalling the relation of ϕ to $\mathcal{R}$. The result for the evolution equation of $\mathcal{R}$ is

$$\mathcal{R}' = \Sigma_{\mathcal{R}} + \frac{\mathcal{H}^2 c_{\mathrm{st}}^2 \nabla^2\beta}{4\pi G a^2 (p_{\mathrm{t}} + \rho_{\mathrm{t}})}, \tag{A.29}$$

where $\Sigma_{\mathcal{R}}$ is defined as

$$\Sigma_{\mathcal{R}} = -\frac{\mathcal{H}\,\delta p_{\mathrm{nad}}}{(p_{\mathrm{t}} + \rho_{\mathrm{t}})} + \frac{\mathcal{H}}{(p_{\mathrm{t}} + \rho_{\mathrm{t}})}\left[\left(c_{\mathrm{st}}^2 - \frac{1}{3}\right)(\delta\rho_{\mathrm{B}} + \delta\rho_{\mathrm{E}}) + \Pi_{\mathrm{E}} + \Pi_{\mathrm{B}}\right]. \tag{A.30}$$

By taking the first derivative of Eq. (A.29), the dependence on $\nabla^2\beta$ can be eliminated using the Hamiltonian and the momentum constraints; the final result is: by means of the other equations we get

$$\mathcal{R}'' + 2\frac{z_{\mathrm{t}}'}{z_{\mathrm{t}}}\mathcal{R}' - c_{\mathrm{st}}^2 \nabla^2\mathcal{R} = \Sigma_{\mathcal{R}}' + 2\frac{z'}{z}\Sigma_{\mathcal{R}} + \frac{3a^4}{z^2}(\Pi_{\mathrm{E}} + \Pi_{\mathrm{B}}), \tag{A.31}$$

where $z_{\mathrm{t}} = (a^2\sqrt{p_{\mathrm{t}} + \rho_{\mathrm{t}}})/(\mathcal{H} c_{\mathrm{st}})$. The variable $z_{\mathrm{t}}\mathcal{R}$ is, up to a sign, the normal mode of an irrotational and relativistic fluid discussed by Lukash [37] (see also [38, 39]) with the difference of the source term containing the dependence on the gauge inhomogeneities. Both Eqs. (A.28) and (A.29) have been discussed in [7]. Note that, from the definition of $\mathcal{R}$ in terms of ϕ and from the Hamiltonian constraint it turns out, as expected, that $\zeta - \mathcal{R} \propto \Delta_\beta$ (see, in particular, the second paper of Ref. [12]). So, with some caveats, the evolution of ζ can be traded from the evolution of $\mathcal{R}$.

B Second-order correlations

The power spectra of the electric and magnetic fields measure the first-order correlation properties of the corresponding fluctuations. The power spectra of the energy densities and of the anisotropic stresses are a measure of the second-order correlation properties of the electric and magnetic fields. To compute the power spectra introduced in Eqs. (3.28)–(3.31) an explicit expression for the magnetic power spectra $P_{\rm B}(q,\tau)$, $P_{\rm E}(q,\tau)$ and $P_{\rm EB}(q,\tau)$ is needed. In an exact de Sitter phase of expansion and for $\mathcal{F} = 2\mathcal{H} = -2/\tau$ the solution of Eqs. (3.8)–(3.10) for the evolution of the power spectra with the correct boundary conditions is given by[17]:

$$P_{\rm B}(k,\tau) = \frac{9 + 3k^2\tau^2 + k^4\tau^4}{4\pi\tau^4}, \qquad P_{\rm E}(k,\tau) = \frac{k^2 + k^4\tau^2}{4\pi^2\tau^2}, \qquad P_{\rm EB}(k,\tau) = -\frac{k(3 + 2k^2\tau^2)}{2\pi^2\tau^3}. \tag{B.1}$$

The same result can be obtained directly from Eq. (3.14) and from the related solutions in terms of the mode functions. In this case the wanted power spectra are:

$$\mathcal{Q}_{\rm B}(q,\tau) = \frac{H^8}{2048\,\pi^7}\int\frac{d^3k}{k^3}\,\frac{q^3}{p^3}[9 + 3k^2\tau^2 + k^4\tau^4][9 + 3p^2\tau^2 + p^4\tau^4]\Lambda_{\rho}(q,k), \tag{B.2}$$

$$\mathcal{Q}_{\rm E}(q,\tau) = \frac{H^8}{2048\,\pi^7}\int d^3k\frac{q^3\,\tau^4}{k\,p}(1 + k^2\tau^2)(1 + p^2\tau^2)\Lambda_{\rho}(q,k), \tag{B.3}$$

$$\mathcal{Q}_{\rm B\Pi}(q,\tau) = \frac{H^8}{4608\,\pi^7}\int\frac{d^3k}{k^3}\,\frac{q^3}{p^3}[9 + 3k^2\tau^2 + k^4\tau^4][9 + 3p^2\tau^2 + p^4\tau^4]\Lambda_{\Pi}(q,k), \tag{B.4}$$

$$\mathcal{Q}_{\rm E\Pi}(q,\tau) = \frac{H^8}{4608\,\pi^7}\int d^3k\frac{q^3\,\tau^4}{k\,p}(1 + k^2\tau^2)(1 + p^2\tau^2)\Lambda_{\Pi}(q,k), \tag{B.5}$$

where $p = |\vec{q} - \vec{k}|$ and the functions $\Lambda_{\rho}(q,k)$ and $\Lambda_{\Pi}(q,k)$ have been defined in Eqs. (3.32)–(3.33). Even if the expressions of Eqs. (B.2)–(B.5) are reasonably simple, it is interesting to bring them to an even simpler (though approximate) form. In particular Eqs. (B.2)–(B.5) are equivalent to the following set of approximate expressions

$$\mathcal{Q}_{\rm B}(q,\tau) = \frac{H^8}{2048\,\pi^7}\biggl[\mathcal{I}_{\rm B}(q)\biggl(\frac{a_{*}}{a}\biggr)^8\vartheta(q - \mathcal{H}) + \mathcal{O}_{\rm B}(q)\vartheta(\mathcal{H} - q)\biggr], \tag{B.6}$$

$$\mathcal{Q}_{\rm E}(q,\tau) = \frac{H^8}{2048\,\pi^7}\biggl[\mathcal{I}_{\rm E}(q)\biggl(\frac{a_{*}}{a}\biggr)^8\vartheta(q - \mathcal{H}) + \mathcal{O}_{\rm E}(q)\biggl(\frac{a_{*}}{a}\biggr)^4\vartheta(\mathcal{H} - q)\biggr], \tag{B.7}$$

$$\mathcal{Q}_{\rm B\Pi}(q,\tau) = \frac{H^8}{4608\,\pi^7}\biggl[\mathcal{I}_{\rm B\Pi}(q)\biggl(\frac{a_{*}}{a}\biggr)^8\vartheta(q - \mathcal{H}) + \mathcal{O}_{\rm B\Pi}(q)\vartheta(\mathcal{H} - q)\biggr], \tag{B.8}$$

$$\mathcal{Q}_{\rm E}(q,\tau) = \frac{H^8}{4608\,\pi^7}\biggl[\mathcal{I}_{\rm E\Pi}(q)\biggl(\frac{a_{*}}{a}\biggr)^8\vartheta(q - \mathcal{H}) + \mathcal{O}_{\rm E\Pi}(q)\biggl(\frac{a_{*}}{a}\biggr)^4\vartheta(\mathcal{H} - q)\biggr], \tag{B.9}$$

where the Heaviside's step function has been introduced. The factorization of the time-dependence has been achieved by expanding the integrands in powers of $k\tau$ and $p\tau$ and by

[17]Recall that during a de Sitter stage of expansion the conformal time coordinate is negative so that all the power spectra of Eq. (B.1) are positive definite.

consistently keeping the leading terms in the expansion. The resulting expressions depend on four integrals over the momenta which can be accurately regularized and computed:

$$
\begin{aligned}
\mathcal{I}_{\mathrm{B}}(q) &= \int d^3k\, k\; p\, q^3\, \Lambda_\rho(q,k), & \mathcal{O}_{\mathrm{B}}(q) &= 81 \int \frac{d^3k}{k^3}\,\frac{q^3}{p^3}\Lambda_\rho(q,k),\\
\mathcal{I}_{\mathrm{E}}(q) &= \int d^3k \frac{q^3\, k^2\, p^2}{k\,p}\Lambda_\rho(q,k), & \mathcal{O}_{\mathrm{E}}(q) &= \int d^3k \frac{q^3}{k\,p}\Lambda_\rho(q,k),\\
\mathcal{I}_{\mathrm{B\Pi}}(q) &= \int d^3k\, k\; p\, q^3\, \Lambda_\Pi(q,k), & \mathcal{O}_{\mathrm{B\Pi}}(q) &= 81 \int \frac{d^3k}{k^3}\,\frac{q^3}{p^3}\Lambda_\Pi(q,k),\\
\mathcal{I}_{\mathrm{E\Pi}}(q) &= \int d^3k \frac{q^3\, k^2\, p^2}{k\,p}\Lambda_\Pi(q,k), & \mathcal{O}_{\mathrm{E\Pi}}(q) &= \int d^3k \frac{q^3}{k\,p}\Lambda_\Pi(q,k).
\end{aligned}
\tag{B.10}
$$

Note that $\mathcal{I}_X(q)$ and $\mathcal{O}_X(q)$ simply denote the modes of the quantity X which are, respectively, inside or outside the Hubble radius at the corresponding epoch as specified by the Heaviside theta functions appearing in Eqs. (B.6)–(B.9).

The energy spectra of the electric and magnetic parts behave differently outside the Hubble radius. While inside the Hubble radius $\mathcal{Q}_{\mathrm{B}}(q,\tau) = \mathcal{Q}_{\mathrm{E}}(q,\tau) \simeq H^8\, a^{-8}$, outside the Hubble radius $\mathcal{Q}_{\mathrm{B}}(q,\tau) \simeq H^8$ is almost constant and $\mathcal{Q}_{\mathrm{E}}(q,\tau) \simeq H^8 a^{-4}$ is sharply decreasing. The same kind of conclusion, with slightly different numerical coefficients, also holds for $\mathcal{Q}_{\mathrm{B\Pi}}(q,\tau)$ and $\mathcal{Q}_{\mathrm{E\Pi}}(q,\tau)$. This means that outside the Hubble radius (which is the most delicate regime from the point of view of the effects on the scalar adiabatic modes), the magnetic components dominate against the electric ones provided the magnetic power spectrum is nearly scale-invariant.

The conclusions drawn so far hold in the case of quantum mechanical initial conditions. This means that the power spectra of the electric and magnetic fields satisfy the corresponding equations for $\mathcal{F} = -2/\tau$ and $\sigma_{\mathrm{c}} = 0$. In the case of conducting initial conditions, the situation is, in some sense, even simpler since electric fields are further suppressed at the level of the initial conditions. This means that, form the relevant equations of the power spectra $P_{\mathrm{EB}}(q,\tau) \simeq 0$ and outside the Hubble radius $\mathcal{O}_{\mathrm{B}}(q,\tau) \simeq H^8 a^{4f-8}$ while $\mathcal{O}_{\mathrm{E}}(q,\tau) \simeq (q/\sigma_{\mathrm{c}})^8 H^8 a^{-4f-8}$ where $f = \mathcal{F}/\mathcal{H}$; $f = 2$ in the case of an exactly scale invariant spectrum.

The spectra of Eqs. (B.2)–(B.5) are derived in the absence of slow roll corrections, i.e. in the case of a pure de Sitter dynamics. In the quasi-de Sitter case, the evolution equations of $f_k(\tau)$ and $g_k(\tau)$ inherit a dependence on the slow roll parameters which enter directly the energy spectra. slow roll corrections are then essential to derive realistic spectra and realistic bounds on the inflationary growth rate of the magnetic inhomogeneities.

For typical wavelengths larger than the Hubble radius the second-order spectra including the slow roll corrections are given by:

$$
\mathcal{Q}_{\mathrm{B}}(q,\tau) = \mathcal{O}_{\mathrm{B}}(q,\epsilon,f)\left(\frac{a}{a_{ex}}\right)^{g_B(\epsilon,f)}, \quad \mathcal{Q}_{\mathrm{B\Pi}}(k,\tau) = \mathcal{O}_{\mathrm{B\Pi}}(q,\epsilon,f)\left(\frac{a}{a_{ex}}\right)^{g_B(\epsilon,f)}, \tag{B.11}
$$

$$
\mathcal{Q}_{\mathrm{E}}(q,\tau) = \mathcal{O}_{\mathrm{E}}(q,\epsilon,f)\left(\frac{a}{a_{ex}}\right)^{g_E(\epsilon,f)}, \quad \mathcal{Q}_{\mathrm{E\Pi}}(q,\tau) = \mathcal{O}_{\mathrm{E\Pi}}(q,\epsilon,f)\left(\frac{a}{a_{ex}}\right)^{g_E(\epsilon,f)}, \tag{B.12}
$$

where $g_B(\epsilon, f)$ and $g_E(\epsilon, f)$ are:

$$g_B(\epsilon, f) = 4f - 8 + 4\epsilon\, f, \qquad g_E(\epsilon, f) = 4f - 12 + 2f\,\epsilon. \tag{B.13}$$

The amplitudes appearing in Eqs. (B.11) and (B.12) are:

$$\mathcal{O}_X(q, \epsilon, f) = H^8\, \mathcal{C}_X(f,\epsilon)\, \mathcal{L}_X(f,\epsilon,q) \biggl(\frac{q}{q_{\mathrm{p}}}\biggr)^{m_X(\epsilon,f)-1} \tag{B.14}$$

where X coincides either with the magnetic (i.e. B, BΠ) or with the electric (i.e. E, EΠ) labels. In the parametrization of Eq. (B.14) the flat spectrum of the X power spectrum arises for $m_X = 1$ and the various indices corresponding to the four components are given by:

$$m_{\mathrm{B}}(\epsilon, f) = m_{\mathrm{B\Pi}}(\epsilon, f) = 9 - 4f(1+\epsilon), \qquad m_{\mathrm{E}}(\epsilon, f) = m_{\mathrm{E\Pi}}(\epsilon, f) = 13 - 4f(1+\epsilon). \tag{B.15}$$

The functions $\mathcal{C}_X(f,\epsilon)$ are given, respectively, by:

$$\mathcal{C}_{\mathrm{B}}(f,\epsilon) = \frac{2^{4f(1+\epsilon)}}{1024\,\pi^7}\,\Gamma^4[f(1+\epsilon)+1/2], \quad \mathcal{C}_{\mathrm{B\Pi}}(f,\epsilon) = \frac{4}{9}\mathcal{C}_{\mathrm{B}}(f,\epsilon), \tag{B.16}$$

$$\mathcal{C}_{\mathrm{E}}(f,\epsilon) = \frac{2^{4f(1+\epsilon)-2}}{4096\,\pi^7}\,\Gamma^4[f(1+\epsilon)-1/2], \quad \mathcal{C}_{\mathrm{E\Pi}}(f,\epsilon) = \frac{4}{9}\mathcal{C}_{\mathrm{E}}(f,\epsilon). \tag{B.17}$$

The functions $\mathcal{L}_X(f,\epsilon,q)$ are:

$$\begin{aligned}
\mathcal{L}_{\mathrm{B}}(f,\epsilon,q) &= \frac{8[f(1+\epsilon)+1]}{3[4f(1+\epsilon)-5][4-2f(1+\epsilon)]} - \frac{8}{3[4-2f(1+\epsilon)]}\biggl(\frac{q}{q_0}\biggr)^{2f(1+\epsilon)-4} \\
&+ \frac{4}{5-4f(1+\epsilon)}\biggl(\frac{q}{q_{\mathrm{max}}}\biggr)^{4f(1+\epsilon)-5}, &\text{(B.18)}\\
\mathcal{L}_{\mathrm{B\Pi}}(f,\epsilon,q) &= \frac{2[17-2f(1+\epsilon)]}{15[4f(1+\epsilon)-5][4-2f(1+\epsilon)]} - \frac{2}{3[4-2f(1+\epsilon)]}\biggl(\frac{q}{q_0}\biggr)^{2f(1+\epsilon)-4} \\
&+ \frac{7}{5-4f(1+\epsilon)}\biggl(\frac{q}{q_{\mathrm{max}}}\biggr)^{4f(1+\epsilon)-5}, &\text{(B.19)}\\
\mathcal{L}_{\mathrm{E}}(f,\epsilon,q) &= \frac{8f(1+\epsilon)}{3[6-2f(1+\epsilon)][4f(1+\epsilon)-9]} - \frac{8}{3[6-2f(1+\epsilon)]}\biggl(\frac{q}{q_0}\biggr)^{2f(1+\epsilon)-6} \\
&+ \frac{4}{9-4f(1+\epsilon)}\biggl(\frac{q}{q_{\mathrm{max}}}\biggr)^{4f(1+\epsilon)-9}, &\text{(B.20)}\\
\mathcal{L}_{\mathrm{E\Pi}}(f,\epsilon,q) &= \frac{2[18-f(1+\epsilon)]}{15[4f(1+\epsilon)-9][6-4f(1+\epsilon)]} - \frac{2}{3[6-2f(1+\epsilon)]}\biggl(\frac{q}{q_0}\biggr)^{2f(1+\epsilon)-6} \\
&+ \frac{7}{5[9-4f(1+\epsilon)]}\biggl(\frac{q}{q_{\mathrm{max}}}\biggr)^{4f(1+\epsilon)-9}. &\text{(B.21)}
\end{aligned}$$

The comoving scale $q_{\mathrm{p}} = 0.002\,\mathrm{Mpc}^{-1}$ is the usual pivot scale at which the power spectra of the scalar curvature are assigned. The value of q_0 has been chosen $0.001\,q_{\mathrm{p}}$ while q_{max} can

be estimated from the transition scale between inflation and radiation and it is of the order of $10^{24}\,(\epsilon \mathcal{A}_{\mathcal{R}})^{1/4}$ Mpc^{-1}. The results reported in Eqs. (B.16)–(B.21) follow after lengthy but straightforward algebra from Eqs. (3.28)–(3.31). Consider, for instance, the second-order correlations of the magnetic energy density. From Eq. (3.28) the explicit expression of $\mathcal{Q}_{\mathrm{B}}(q,\tau)$ can be written as:

$$\mathcal{Q}_{\mathrm{B}}(q,\tau) = \frac{H^8}{8192\pi^7}\int_{-1}^{1} dy \int_{u_0}^{u_{\max}} \frac{du}{u}\, s^3\, u^5\, |\vec{s}-\vec{u}|\, |H_{\nu}^{(1)}(u)|^2\, |H_{\nu}^{(1)}(|\vec{s}-\vec{u}|)|^2\, \Lambda_{\rho}(u,s,y), \quad \text{(B.22)}$$

where $y = \cos\vartheta$ is one of the angular variables arising from the integration over the comoving three-momentum and where the following dimesnionless vectors have been introduced

$$\vec{s} = \frac{\vec{q}}{aH}, \qquad \vec{u} = \frac{\vec{k}}{aH}, \qquad |\vec{s}-\vec{u}| = \frac{|\vec{q}-\vec{k}|}{aH}. \quad \text{(B.23)}$$

In Eq. (B.22) $H_{\nu}^{(1)}(z)$ denotes the Hankel function (of generic argument z) coming from the solution of the mode equations including the slow roll corrections. In a specific model, such as the ones discussed in Sec. 5, $\mathcal{F}$ will assume a specific dependence on the scale factor and we shall focus on the case of a monotonic dependence. The Bessel index ν of Eq. (B.23) will then depend both on f and on the slow roll parameter. This happens since the mode equation for $f_k(\tau)$ (which is the one relevant for Eqs. (3.28) and (B.22)) can be written as[18]

$$f_k'' + [k^2 - \mathcal{F}^2 - \mathcal{F}']f_k = 0, \qquad \mathcal{F}^2 + \mathcal{F}' = a^2 H^2 [f^2 + f(1+\epsilon)]. \quad \text{(B.24)}$$

In the present investigation we preferentially considered models, compatible with the conventional inflationary scenario, where λ depends on a spectator field and it slowly increases during the quasi-de Sitter stage at a rate which we ought to constrain. If the slow roll parameters are all constant (as it happens in the case of monomial inflationary potentials, for instance) then aH is given by Eq. (2.20) and, to first order in ϵ, $\nu \simeq f + 1/2 + f\epsilon$. The integration over y in the class of integrals represented by Eq. (B.22) can be performed explicitly, after some algebra, when the given wavelengths are either larger or smaller than the Hubble radius. In connection with the lengthy algebra, Eqs. (3.32)–(3.33) imply that, in Eq. (B.23), $\Lambda_{\rho}(u,s,y)$ depends on $y = \cos\vartheta$; the same holds for $\Lambda_{\Pi}(u,s,y)$ in the other integrals involving electric and magnetic anisotropic stresses. Using this strategy all the explicit expressions reported in Eqs. (B.18)–(B.20) can be obtained after radial integration.

[18]Note that the mode function $f_k(\tau)$ cannot be confused with f the wavenumber has been always written explicitly. With this caveat potential confusions are avoided.

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